

On The Existence of Min-Max Minimal Surface of Genus $g \geq 2$

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Abstract

We prove an existence theorem similar to that of [5][15] for min-max minimal surfaces of genus $g \geq 2$ by variational methods. We show that the min-max critical value for the area functional can be achieved by the bubbling limit of branched minimal surfaces with nodes of genus g together with possibly finitely many branched minimal spheres. We also prove a strong convergence theorem similar to the classical mountain pass lemma. It is a further extension of the existence result in [5][15].

1 Introduction

Existence theory of minimal surfaces and application to geometry and topology have been studied for a long time since the proof of classical Plateau Problem (see Chapter 4 of [3]) in 1931. There are lots of interesting results concerning the existence of area minimizing minimal surfaces in a given homotopy class. In particular, the existence theory for area-minimizing surfaces has been developed and applied to study topology for all genus in suitable senses (cf. [12][13][14][6] etc.). Recently, existence theory for min-max minimal surfaces has attracted more interest, and has had nice applications and potential significance (cf. [2][4][5][11]). One remarkable work was given by T. Colding and W. Minicozzi in [4][5], where they constructed min-max minimal spheres and proved the finite time extinction for Ricci flow under certain topological conditions by studying the evolution of the area of the min-max minimal spheres. Motivated by that paper [5] the author studied the variational construction of min-max minimal tori in [15]. The

difference between spheres and surfaces of genus greater than zero is that the moduli space of conformal structures is nontrivial. The author developed a uniformization result in [15] to deal with this difficulty in the case of tori¹.

High genus cases are always very interesting (see [13][14] for application in the minimizing case). Recently, F. Marques and A. Neves gave an application of the min-max theory in the geometric measure theory setting(see [2]) to get some very interesting rigidity results on positive curved compact manifold. In this paper, we will extend the result in [5][15] to the high genus case($g \geq 2$). Since the moduli space of surfaces of genus larger than one is more complicated than that of the tori(genus one), we need a more delicate uniformization result. Besides this, we also need to extend the local replacement method and bubbling convergence theory in [5] to this case. Using notations introduced in Section 2.2, we summarize our main theorem as:

Theorem 1.1 *For any homotopically nontrivial path $\beta \in \Omega$, if $\mathcal{W} > 0$, there exists a sequence $(\rho_n, \tau_n) \in [\beta]$, with $\max_{t \in [0,1]} E(\rho_n(t), \tau_n(t)) \rightarrow \mathcal{W}$, and for any $\epsilon > 0$, there exists a large number $N > 0$ and a small number $\delta > 0$ such that if $n > N$, then for any $t \in (0, 1)$ satisfying:*

$$E(\rho_n(t), \tau_n(t)) > \mathcal{W} - \delta, \quad (1)$$

there are a conformal harmonic map $u_0 : \bar{\Sigma}_g \rightarrow N$ defined on the body of a genus g Riemann surface with nodes Σ_g^ and possibly finitely many harmonic sphere $u_i : S^2 \rightarrow N$, such that:*

$$d_V(\rho_n(t), \cup_i u_i) \leq \epsilon. \quad (2)$$

Here the definition of Riemann surfaces with nodes is given in Section 5.1, and d_V means varifold distance given in Appendix A in [5]. The theorem follows from Theorem 5.1 and Appendix A of [5].

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¹In the case of tori, [7] also gave a method to deal with moduli space in an evolution setting.

2 Sketch of the variational methods

Now let us firstly review the method used by the author in [15]. In this method, we consider the area functional and energy functional simultaneously. Let (N, h) be the target manifold. Consider the space of paths² $\Omega = \left\{ \gamma(t) \in C^0\left([0, 1], C^0 \cap W^{1,2}(T_0^2, N)\right) \right\}$, where a path is a one parameter family of mappings $\gamma(t)$ from tori to the target manifold. If we add certain degeneration constrains on the ends of all the paths, i.e. $\gamma(0), \gamma(1)$ are constant maps or maps to closed curves in N , we can define a min-max critical value $\mathcal{W} = \inf_{\rho \in [\beta]} \max_{t \in [0, 1]} \text{Area}(\rho(t))$ on a homotopy class of $\beta(t) \in \Omega$, where $\text{Area}(\cdot)$ is the area functional. Suppose the critical value is positive, i.e $\mathcal{W} > 0$. A natural question is how to find the corresponding critical points. Classical 2-dimensional geometric variational methods are used to find the critical points. Firstly and naturally, take an area minimizing sequence of paths $\tilde{\gamma}_n(t) \in [\rho]$, such that $\lim_{n \rightarrow \infty} \max_{t \in [0, 1]} \text{Area}(\tilde{\gamma}_n(t)) = \mathcal{W}$. Then we need to use the energy functional. Since energy functional depends not only on the mappings, but also on the conformal structures of the domain metrics, we need to module out the action of conformal group. We consider the following min-max critical value³ $\mathcal{W}_E = \inf_{(\rho, \tau) \in [(\beta, \tau_0)]} \max_{t \in [0, 1]} E(\rho(t), \tau(t))$, where E is the energy functional. In fact, $\mathcal{W}_E = \mathcal{W}$ (See Section 3 in [15]). In order to module out conformal group action, we need to do reparametrizations on tori. Let $\tilde{g}_n(t) = \tilde{\gamma}_n(t)^* h$ be the pullback of the ambient metric, which may be degenerate. Using a uniformization result proved in [15] and a perturbation technique, $\tilde{g}_n(t)$ determines a continuous family of elements $\tau_n(t)$ in the Teichmüller space \mathcal{T}_1 of genus one and a continuous isotopy family of diffeomorphisms $h_n(t) : (T^2, \tau_n(t)) \rightarrow (T^2, \tilde{g}_n(t))$, and if denoting $\gamma_n(t) = \tilde{\gamma}_n(t) \circ h_n(t)$, $E(\gamma_n(t), \tau_n(t)) - \text{Area}(\gamma_n(t)) \rightarrow 0$. After that, we do compactification to the sequences of path $\gamma_n(t)$ by the method of harmonic replacement developed by Colding and Minicozzi in [5]. Lastly, we combine the degeneration of conformal structures with the bubbling convergence⁴ firstly developed by Sack and Uhlenbeck in [12] to give a combined bubbling convergence for the compactified sequence of paths (See Theorem 5.1 in [15]). In the limit, we get some conformal harmonic maps from torus together with possibly some spheres or only harmonic spheres when degeneration happens. We can

²They are also called sweep-outs in [5].

³See [15] for details of the notations.

⁴See also [5][7] for bubble convergence.

also get the energy identity (equation (45), (16) in [15]). In fact, we will achieve a strong deformation results for this specialized sequences (i.e. Theorem 1.1 of [15]).

Based on this method, let us describe the approach to high genus cases.

2.1 Teichmüller spaces of genus g surfaces

Before going into the variational method, let us firstly describe the Teichmüller spaces \mathcal{T}_g and moduli spaces \mathcal{M}_g of a genus g surface Σ_g . We will summarize the following facts about Teichmüller spaces.

- 1° : Definition about Teichmüller spaces and Moduli spaces;
- 2° : Marked surface representation of Teichmüller spaces;
- 3° : Fuchsian model description for Teichmüller spaces;
- 4° : Quasi-conformal maps;
- 5° : Teichmüller mappings;

1°. Let Met_g be the space of all the metrics on a topological surface Σ_g of genus g . Denote $Diff(\Sigma_g)$ by the self diffeomorphism groups on Σ_g , and $Diff_0(\Sigma_g)$ the subgroup of $Diff(\Sigma_g)$ containing elements isotopy to the identity. Two metrics ds^2 and $(ds^2)'$ are said to be equivalent in the sense of moduli space, if there exists $w \in Diff(\Sigma_g)$, such that $w^*(ds^2)'$ is conformal to ds^2 . Define all the equivalent classes to be the *moduli space* $\mathcal{M}_g = Met_g / Diff(\Sigma_g)$. Two metrics ds^2 and $(ds^2)'$ are said to be equivalent in the sense of Teichmüller space, if there exists $w \in Diff_0(\Sigma_g)$, such that $w^*(ds^2)'$ is conformal to ds^2 . Define all the equivalent classes to be the *Teichmüller space* $\mathcal{T}_g = Met_g / Diff_0(\Sigma_g)$. Here we are interested in the complex structure of the surfaces, obviously each (Σ_g, ds^2) has a complex structure compatible with ds^2 . Later on, we will use this complex structure without mentioning it.

2°. We firstly talk about the representation of Teichmüller spaces by the marked surfaces. We use the description in [9]. Given a fixed genus g surface Σ_0 , consider all the surfaces (Σ, f) , with $f : \Sigma_0 \rightarrow \Sigma$ a diffeomorphism. We say that (Σ, f) and (Σ', g) are equivalent in the sense of Teichmüller space, if $g \circ f^{-1} : \Sigma \rightarrow \Sigma'$ is homotopic to a conformal diffeomorphism from Σ to Σ' . We call such a f a *marking*, and (Σ, f) a *marked surface*. The set of all equivalent classes of marked surfaces $\{[(\Sigma, f)]\}$ is another representation of the Teichmüller spaces \mathcal{T}_g of genus g (Chap 1 of [9]).

3°. Let us talk about the Fuchsian model now. By uniformization theorem from complex analysis, all the closed surfaces Σ_g with genus $g > 1$ have their universal covering space the upper half plane \mathbb{H} . The covering transformation group of $\pi : \mathbb{H} \rightarrow \Sigma_g$

is called *Fuchsian group*, which is denoted by Γ , and (Σ_g, Γ) is called *Fuchsian model*. Usually, we also simply call Γ a Fuchsian model. In the sense of complex analysis, the holomorphic diffeomorphism group of \mathbb{H} is $PSL(2, \mathbb{R})$, so Γ contains only linear fractional transformations with real coefficients, i.e, $\Gamma \subset PSL(2, \mathbb{R})$. If we consider the hyperbolic metric structure (\mathbb{H}, ds_{-1}^2) , where $ds_{-1}^2 = \frac{dx^2 + dy^2}{y^2}$, Γ is constituted by isometries of (\mathbb{H}, ds_{-1}^2) .

4°. We also need to talk about the quasi-conformal maps. Let $f : \Sigma \rightarrow \Sigma'$ be a diffeomorphism between two Riemann surfaces. Given local complex coordinates (z, \bar{z}) , (w, \bar{w}) on Σ and Σ' respectively. Denote $f(z) = w \circ f \circ z$. Let

$$\mu = \frac{f_{\bar{z}}}{f_z}. \quad (3)$$

It is easy to see that $|\mu|$ does not depend on the local complex coordinates. If $|\mu| \leq k < 1$, then we call such f a *quasi-conformal map*⁵.

Now let us combine the marked surface model with the quasi-conformal maps (see 5.1.2 of [9]). Suppose Σ_0 is a fixed Riemann surface, with a Fuchsian group Γ_0 . After some conjugation in $PSL(2, \mathbb{R})$, we can always assume $(0, 1, \infty)$ are fixed by some elements in Γ_0 (see chap 5 in [9]). We call such Γ_0 a *normalized Fuchsian group*, and (Σ_0, Γ_0) a *normalized Fuchsian model*. For any marked surface (Σ, f) , $f : \Sigma_0 \rightarrow \Sigma$ is always a quasi-conformal map (1.4.2 of [9]). Now we lift the quasi-conformal map f up to the upper half space \mathbb{H} by the covering maps $\pi_0 : \mathbb{H} \rightarrow \Sigma_0$ and $\pi : \mathbb{H} \rightarrow \Sigma$ to get $\tilde{f} : \mathbb{H} \rightarrow \mathbb{H}$. After some $PSL(2, \mathbb{R})$ action on the target \mathbb{H} , we can assume that \tilde{f} also fixes the three points $(0, 1, \infty)$ (We know the uniqueness of such quasi-conformal maps from [9] and also from the following discussion). We call such maps $\tilde{f} : \mathbb{H} \rightarrow \mathbb{H}$ *canonical quasi-conformal maps*. By pulling over the normalized Fuchsian group Γ_0 on Σ_0 by \tilde{f} , we get another Fuchsian group $\Gamma_{\tilde{f}} = \tilde{f} \circ \Gamma_0 \circ \tilde{f}^{-1}$, such that $\Sigma = \mathbb{H} / \Gamma_{\tilde{f}}$. Now for such a marking f , we can define an injective homeomorphism:

$$\theta_{\tilde{f}} : \Gamma_0 \rightarrow PSL(2, \mathbb{R}),$$

where $\theta_{\tilde{f}}(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$. Lemma 5.1 in [9] tells that (Σ_1, f_1) and (Σ_2, f_2) are equivalent in the sense of Teichmüller space, iff $\theta_{\tilde{f}_1} = \theta_{\tilde{f}_2}$. Now we can define the following set:

$$\mathcal{T}_g^\sharp = \left\{ \theta_{\tilde{f}} : \begin{array}{l} \tilde{f} \text{ is a canonical quasiconformal map, such that} \\ \theta_{\tilde{f}}(\Gamma_0) \text{ is a Fuchsian group for some genus } g \text{ surface.} \end{array} \right\} \quad (4)$$

⁵When $|\mu| = 0$, f is holomorphic.

Proposition 5.3 of [9] shows that \mathcal{T}_g^\sharp is identified with the Teichmüller space \mathcal{T}_g . Later on, we will use this representation of Teichmüller space of genus g , and we will extend the quasi-conformal maps to more general settings, say, in the Sobolev spaces.

5°. We also need to talk about the Teichmüller mapping in a class of marked surfaces $[(\Sigma, f)]$, where $f : \Sigma_0 \rightarrow \Sigma$ is a diffeomorphism, and also a quasi-conformal map. By Theorem 5.9 in [9], there exists a unique holomorphic quadratic differential ϕ on Σ_0 with $\|\phi\|_1 < 1^6$, and a unique quasi-conformal mapping $f_1 : \Sigma_0 \rightarrow \Sigma$ homotopic to f , and that the Beltrami coefficient μ_{f_1} of f_1 satisfies $\mu_{f_1} = \mu_\phi$, where

$$\mu_\phi \equiv \|\phi\|_1 \frac{\bar{\phi}}{|\phi|}. \quad (5)$$

We denote such a map by f_ϕ and call it *Teichmüller mapping*.

Denote the set of all holomorphic quadratic differentials on Σ_0 with one-norm $\|\cdot\|_1$ strictly less than one by $A_2(\Sigma_0)_1$. Given $\phi \in A_2(\Sigma_0)_1$, let f_ϕ be the unique Teichmüller mapping of the Beltrami coefficient μ_ϕ in the homotopy class $id : \Sigma_0 \rightarrow \Sigma_0$. From Theorem 5.9 in [9], we know that the mapping

$$\mathcal{F} : A_2(\Sigma_0)_1 \rightarrow \mathcal{T}_g^\sharp,$$

is a homeomorphism, where $\mathcal{F}(\phi) = \theta_{\tilde{f}_\phi}$ and $\tilde{f}_\phi : \mathbb{H} \rightarrow \mathbb{H}$ is the unique canonical quasi-conformal mapping lifted up of f_ϕ . By Riemann-Roch theorem, we know that $A_2(\Sigma_0)_1$ is homotopic to a $6g - 6$ dimension ball, and hence is \mathcal{T}_g^\sharp . Later on, the topology on \mathcal{T}_g and \mathcal{T}_g^\sharp is identified by the topology on $A_2(\Sigma_0)_1$.

2.2 Some notation

Now let us set down the framework of the variational method. Given a Riemannian manifold (N, h) . Let Σ_0 be a fixed Riemann surface of genus $g > 1$ with a normalized Fuchsian group Γ_0 . Denote elements in Teichmüller space \mathcal{T}_g by τ . Let $\phi_\tau \in A_2(\Sigma_0)_1$ be the unique holomorphic quadratic differential on Σ_0 corresponding to τ . Denote $f_\tau = f_{\phi_\tau}$ to be the unique Teichmüller mapping determined by the Beltrami coefficient μ_{ϕ_τ} , and $\tilde{f}_\tau : \mathbb{H} \rightarrow \mathbb{H}$ the unique canonical quasi-conformal mapping lifted up with respect to Γ_0 . By the results in the above section, we can view τ as an equivalent class of marked surfaces $[(\Sigma_\tau, f_\tau)]$ with normalized Fuchsian group $\Gamma_\tau = \theta_{\tilde{f}_\tau}(\Gamma_0)$, i.e. $\Sigma_\tau = \mathbb{H}/\Gamma_\tau$.

⁶Here $\|\phi\|_1$ is the L^1 norm of ϕ .

Definition 2.1 *The variational spaces are defined as*

$$\Omega = \{ \gamma(t) \in C^0([0, 1], C^0 \cap W^{1,2}(\Sigma_0, N)) \}, \quad (6)$$

and

$$\tilde{\Omega} = \{ (\gamma(t), \tau(t)) : \gamma(t) \in C^0([0, 1], C^0 \cap W^{1,2}(\Sigma_{\tau(t)}, N)), \tau(t) \in C^0([0, 1], \mathcal{T}_g) \}, \quad (7)$$

where $(\Sigma_\tau = \mathbb{H}/\Gamma_\tau, \Gamma_\tau)$ is the normalized Fuchsian model corresponding to $\tau \in \mathcal{T}_g$. We always assume that the boundary mappings $\gamma(0)$ and $\gamma(1)$ are mapped onto close curves in N .

Now let us talk about the continuity of $\gamma(t) \in C^0([0, 1], C^0 \cap W^{1,2}(\Sigma_{\tau(t)}, N))$. Here we can view all the $\gamma(t)$ as been defined on the upper half plane \mathbb{H} lifted up by $\pi_{\tau(t)} : \mathbb{H} \rightarrow \Sigma_{\tau(t)}$, with the Fuchsian groups $\Gamma_{\tau(t)}$ changing continuously w.r.t the parameter t . The continuity of $\gamma(t)$ w.r.t parameter t can be defined as mappings on compact subsets K of \mathbb{H} with the Poincaré metric, i.e. $\gamma(t) \in C^0([0, 1], C^0 \cap W^{1,2}(K, N))$. Another way to understand this is as follows. Let $\phi_{\tau(t)}$ be the holomorphic quadratic differentials corresponding to $\tau(t)$. The fact that $\tau(t)$ change continuously w.r.t t is equivalent to that $\phi_{\tau(t)}$ change continuously w.r.t t in $A_2(\Sigma_0)_1$. Let $f_{\tau(t)}$ be the Teichmüller mappings corresponding to $\phi_{\tau(t)}$, then the canonical lift $\tilde{f}_{\tau(t)} : \mathbb{H} \rightarrow \mathbb{H}$ change continuously in $C_{loc}^0 \cap W^{1,2}(\mathbb{H}, \mathbb{H})$ by properties of quasi-conformal mapping.⁷ Using $f_{\tau(t)}$ as special markings for a continuous family of elements in \mathcal{T}_g , we can pull the path $\gamma(t) : \Sigma_{\tau(t)} \rightarrow N$ back to Σ_0 , i.e. $f_{\tau(t)}^*(\gamma(t)) = \gamma(t) \circ f_{\tau(t)} : \Sigma_0 \rightarrow N$. The continuity of $\gamma(t)$ w.r.t t is defined as the continuity of the path $f_{\tau(t)}^*\gamma(t)$ w.r.t t on the same domain surface Σ_0 .

Next let us talk about the homotopy equivalence in $\tilde{\Omega}$. Consider two elements $\{(\gamma_i(t), \tau_i(t)) : i = 1, 2\}$. They have different domains $\Sigma_{\tau_i(t)}$, $i = 1, 2$ given by normalized Fuchsian models $\Gamma_{\tau_i(t)}$. As above, we use Teichmüller mappings $f_{\phi_{\tau_i(t)}} : \Sigma_0 \rightarrow \Sigma_{\tau_i(t)}$, $i = 1, 2$ to identify $\Sigma_{\tau_i(t)}$, $i = 1, 2$ with Σ_0 , where $\phi_{\tau_i(t)}$ are the holomorphic quadratic differentials corresponding to $\tau_i(t)$, $i = 1, 2$. Since \mathcal{T}_g is homotopic to a ball, $\tau_1(t)$ and $\tau_2(t)$ are always homotopic to each other. So $\{(\gamma_1(t), \tau_1(t))\}$ are homotopic to $\{(\gamma_2(t), \tau_2(t))\}$ if $f_{\phi_{\tau_1(t)}}^*\gamma_1(t)$ are homotopic to $f_{\phi_{\tau_2(t)}}^*\gamma_2(t)$.

Definition 2.2 *Fix a homotopy class $[\beta] \subset \Omega$, and τ_0 a fixed element in \mathcal{T}_g given by $[(\Sigma_0, id)]$. For area functional, define*

$$\mathcal{W} = \inf_{\rho \in [\beta]} \max_{t \in [0, 1]} \text{Area}(\rho(t)). \quad (8)$$

⁷See Chap 4 of [9] and the following Section.

For energy functional, define

$$\mathcal{W}_E = \inf_{(\rho, \tau) \in [(\beta, \tau_0)]} \max_{t \in [0, 1]} E(\rho(t), \tau(t)). \quad (9)$$

Remark 2.1 The definition of area and energy is referred to [10][12][5]. Later, we will show that $\mathcal{W} = \mathcal{W}_E$ in Remark 3.2. We will mainly focus on the case when $\mathcal{W} > 0$.

2.3 Sketch of the variational method

Now an interesting question is to find the critical points corresponding to \mathcal{W} . In fact, the critical points are achieved by some conformal harmonic mappings from surfaces degenerated from Σ_0 together with possibly some harmonic spheres. To achieve the critical points, we use geometric variational method. We take a minimizing sequence $\{\tilde{\gamma}_n(t) : n = 1, \dots, \infty\} \subset [\beta] \subset \Omega$, such that

$$\lim_{n \rightarrow \infty} \max_{t \in [0, 1]} \text{Area}(\tilde{\gamma}_n(t)) = \mathcal{W}.$$

In fact, by some mollification method, we can assume that $\tilde{\gamma}_n(t)$ varies continuously in C^2 -class, i.e. $\tilde{\gamma}_n(t) \in C^0([0, 1], C^2(\Sigma_0, N))$.

Then we would like to change to use the variational method of the energy functional E and hence work in $\tilde{\Omega}$. We use the following three steps. **Firstly**, we do almost conformal reparametrizations to module out the conformal group action. Pull back the ambient metric $\tilde{g}_n(t) = \tilde{\gamma}_n(t)^* h$. We want to show that $\tilde{g}_n(t)$, which may be degenerate, determine a family of elements $\tau_n(t) \in \mathcal{T}_g$. Suppose that the corresponding normalized Fuchsian model and Teichmüller mappings are $(\Sigma_{\tau_n(t)}, \Gamma_{\tau_n(t)}, f_{\tau_n(t)})$, with $\Gamma_{\tau_n(t)} = \theta_{\tilde{f}_{\tau_n(t)}}(\Gamma_0)$ and $\Sigma_{\tau_n(t)} = \mathbb{H}/\Gamma_{\tau_n(t)}$. We want to find almost conformal parametrizations $h_n(t) : \Sigma_{\tau_n(t)} \rightarrow (\Sigma_0, \tilde{g}_n(t))$, such that the reparametrization $(\gamma_n(t), \tau_n(t)) = (\tilde{\gamma}_n(h_n(t), t), \tau_n(t)) \in [(\tilde{\gamma}_n(t), \tau_0)]$ have energy close to area, i.e. $E(\gamma_n(t), \tau_n(t)) - \text{Area}(\gamma_n(t)) \rightarrow 0$. **Secondly**, we do compactification by deforming $\gamma_n(t)$ to $\rho_n(t)$. We use the local harmonic replacement method developed by Colding and Minicozzi in [5][15]. We make $\rho_n(t)$ to be almost harmonic mappings to get bubbling compactness as in [12][5][15]. **Finally**, we discuss the degenerations of conformal structures of $\tau_n(t)$. We will show that $(\rho_n(t), \tau_n(t))$ bubbling converge to several conformal harmonic mappings on surfaces degenerated from Σ_0 together with possibly some harmonic spheres, and we will show that the sum of the area⁸ is equal to \mathcal{W} .

In the following sections, we will discuss the three steps in details.

⁸This is also the energy since the final targets are all conformal.

3 Conformal parametrization in the high genus case

In this section, we will do almost conformal reparametrization for the minimizing sequence $\tilde{\gamma}_n(t) \in \Omega$. We can assume that $\tilde{\gamma}_n(t)$ have better regularity.

Lemma 3.1 (*Lemma D.1 of [5], Lemma 3.1 of [15]*) Suppose $\tilde{\gamma}_n(t)$ are chosen as in the above section, we can perturb them to get a new minimizing sequence in the same homotopy class $[\beta]$, such that if denoting them still as $\tilde{\gamma}_n(t)$, $\tilde{\gamma}_n(t) \in C^0([0, 1], C^2(\Sigma_0, N))$.

3.1 Summary of results on quasi-conformal mappings

Before going to the uniformization and reparametrization, we firstly summarize results of quasi-conformal mappings proved in [1][9] and the appendix of [15]. We will focus on the apriori estimates for the conformal diffeomorphisms between general metrics.

3.1.1 Results about quasi-conformal maps

We mainly refer to Ahlfors and Bers in [1](see also Section 6.1 in [15]). They gave the **existence** and **uniqueness** of conformal diffeomorphism $f^\mu : \mathbb{C}_{|dz+\mu d\bar{z}|^2} \rightarrow \mathbb{C}_{dwd\bar{w}}^9$ fixing three points $(0, 1, \infty)$ for any L^∞ function μ with $|\mu| \leq k < 1$ (see also Theorem 4.30 and Proposition 4.33 of [9]). We can such μ Beltrami coefficient here¹⁰. Such maps satisfy the following equation (see equation 57):

$$f_{\bar{z}}^\mu = \mu(z) f_z^\mu. \quad (10)$$

Define function space $B_p(\mathbb{C}) = C^{1-\frac{2}{p}} \cap W_{loc}^{1,p}(\mathbb{C})$, where $p > 2$ depends on the bound k of $|\mu|$. Suppose $\mu, \nu \in L^\infty(\mathbb{C})$, and $|\mu|, |\nu| \leq k$, with $k < 1$. Let f^μ, f^ν be the corresponding conformal homeomorphisms, then:

Lemma 3.2 (*Lemma 16, Theorem 7, Lemma 17, Theorem 8 of [1], Lemma 6.2 of [15]*)

$$d_{S^2}(f^\mu(z_1), f^\mu(z_2)) \leq c d_{S^2}(z_1, z_2)^\alpha, \quad (11)$$

$$\|f_z^\mu\|_{L^p(B_R)} \leq c(R), \quad (12)$$

⁹We use $\{z, \bar{z}\}$ and $\{w, \bar{w}\}$ as complex coordinates.

¹⁰This is different from that in Section 2.2, without invariance under Fuchsian group

$$d_{S^2}(f^\mu(z), f^\nu(z)) \leq C\|\mu - \nu\|_\infty, \quad (13)$$

$$\|(f^\mu - f^\nu)_z\|_{L^p(B_R)} \leq C(R)\|\mu - \nu\|_\infty. \quad (14)$$

Here d_{S^2} is the sphere distance, which is equivalent to the plane distance of \mathbb{C} on compact sets. $\alpha = 1 - \frac{2}{p}$. B_R is a disk of radius R on \mathbb{C} . All constants are uniformly bounded depending on $k < 1$.

3.1.2 Results about quasi-linear quasi-conformal maps

What we concern in our case are the conformal homeomorphisms $h^\mu : \mathbb{C}_{dwd\bar{w}} \rightarrow \mathbb{C}_{|dz+\mu d\bar{z}|^2}$ fixing three points $(0, 1, \infty)$, which arise as the inverse mappings of those f^μ of Ahlfors and Bers. In fact, suppose

$$h^\mu(w) = (f^\mu)^{-1}(w), \quad (15)$$

then our mappings satisfy:

$$h^\mu_w = -\mu(h^\mu(w))\overline{h^\mu_w}. \quad (16)$$

Since the equation is quasi-linear (compared to linear equation 10), we call such h^μ *quasi-linear quasi-conformal maps*.

If $\{\mu_n\}$ are a sequence of Beltrami coefficients as above, such that $\|\mu_n - \mu\|_{C^1} \rightarrow 0$, and h^{μ_n} satisfying 15, we have results similar to the above:

Lemma 3.3 (*Lemma 6.3 of [15]*)

$$d_{S^2}(h^{\mu_n}, h^\mu) \rightarrow 0, \quad (17)$$

$$\|(h^{\mu_n} - h^\mu)_w\|_{L^p(B_R)} \rightarrow 0, \quad (18)$$

where p is given in the above section.

3.2 Uniformization for surfaces of genus $g > 1$

Fix Σ_0 with normalized Fuchsian model Γ_0 as before. Denote π_0 to be the quotient map for (Σ_0, Γ_0) . Denote the Poincaré metric on Σ_0 by g_0 . Given $\tau \in \mathcal{T}_g$, let the corresponding normalized Fuchsian model be $(\mathbb{H}, \Gamma_\tau, \Sigma_\tau)$ as in Section 2.2. Let $\pi_\tau : \mathbb{H} \rightarrow \Sigma_\tau$ be the quotient map, and $f_\tau : \Sigma_0 \rightarrow \Sigma_\tau$ the Teichmüller mapping.

Proposition 3.1 *Let g be a C^1 metric on Σ_0 . We can view g as a metric on \mathbb{H} by lifting up using π_0 . Then there is a unique element $\tau \in \mathcal{T}_g$ with normalized Fuchsian model $(\Sigma_\tau, \Gamma_\tau)$, and a unique orientation preserving $C^{1, \frac{1}{2}}$ conformal diffeomorphism $h : \Sigma_\tau \rightarrow (\Sigma_0, g)$, such that h is isotopic to f_τ^{-1} , with the normalization that if lifting up to $\tilde{h} : \mathbb{H} \rightarrow \mathbb{H}$ by π_τ and π_0 , $\tilde{h}^*(\Gamma_0) = \Gamma_\tau$. Furthermore, given $g(t)$ a family of C^1 metrics on Σ_0 which is continuous w.r.t t in the C^1 class, i.e. $g(t) \in C^1([0, 1], C^1\text{-metrics})$, and $g(t) \geq \epsilon g_0$ for some uniform $\epsilon > 0$, let $(\tau(t), h(t))$ be the corresponding elements in \mathcal{T}_g and normalized conformal diffeomorphisms, then $\tau(t)$ and $h(t)$ are continuously w.r.t t in \mathcal{T}_g and $C^0 \cap W^{1,2}(\Sigma_{\tau(t)}, \Sigma_0)$ respectively.*

Remark 3.1 *Here the space $C^0 \cap W^{1,2}(\Sigma_{\tau(t)}, \Sigma_0)$ have varying domain spaces $\Sigma_{\tau(t)}$, and the continuity is defined in Section 2.2.*

We need the following result to prove the proposition. Let g be a Riemannian metric on the complex plane \mathbb{C} .

Lemma 3.4 *(Lemma 6.1 of [15]) In the complex coordinates $\{z, \bar{z}\}$, we can write $g = \lambda(z)|dz + \mu(z)d\bar{z}|^2$. Here $\lambda(z) > 0$, and $\mu(z)$ is complex function on the complex plane with $|\mu| < 1$. If $g \geq \epsilon dzd\bar{z}$, there exists a $k = k(\epsilon) < 1$, such that $|\mu| \leq k$. Furthermore, μ is a rational function of the components $g_{ij}(z)$, so if a family $g(t)$ is continuous w.r.t t in the C^1 class, the corresponding $\mu(t)$ is also continuous in the C^1 class.*

Proof: (of Proposition 3.1). Firstly, let us show the existence of such mark $\tau \in \mathcal{T}_g$ and conformal homeomorphism h . Pulling g back to \mathbb{H} by π_0 and denote it still by g , then it is invariant under the Γ_0 group action. By Lemma 3.4, $g = \lambda(z)|dz + \mu(z)d\bar{z}|^2$, with $|\mu(z)| \leq k < 1$. Here μ is the Beltrami coefficient as in Section 3.1.1. We have a unique normalized quasi-conformal mapping $f^\mu : \mathbb{H}_{|dz + \mu d\bar{z}|^2} \rightarrow \mathbb{H}_{dwd\bar{w}}$ (see also Proposition 4.33 of [9]). Now push forward the Fuchsian group Γ_0 under f^μ . Since f^μ is a homeomorphism, we get another Fuchsian group $\Gamma_{f^\mu} = (f^\mu)_*(\Gamma_0) = \theta_{f^\mu}(\Gamma_0)$ on $\mathbb{H}_{dwd\bar{w}}$. This Fuchsian group gives a normalized Fuchsian model which represent an element in \mathcal{T}_g . Denote this element by τ . Denoting Γ_{f^μ} by Γ_τ , we get a Fuchsian model $\Sigma_\tau = \mathbb{H}/\Gamma_\tau$. Let $\pi_\tau : \mathbb{H} \rightarrow \Sigma_\tau$ be the quotient map, then after taking quotient of f^μ by π_0 and π_τ , we get $f^\mu : \Sigma_0 \rightarrow \Sigma_\tau$ ¹¹. By the definition of quasi-conformal, this f^μ is conformal between $(\Sigma_0, |dz + \mu(z)d\bar{z}|^2)$ and Σ_τ , and hence conformal between (Σ_0, g) and Σ_τ . So we take $h = (f^\mu)^{-1}$, then h is a conformal homeomorphism between Σ_τ

¹¹We denote the quotient map still by f^μ

and (Σ_0, g) . The $C^{1, \frac{1}{2}}$ regularity of h follows from Theorem 3.1.1 and Theorem 3.3.1 in [10]. By the definition of $f_\tau : \Sigma_0 \rightarrow \Sigma_\tau$, when pulling back to $\tilde{f}_\tau : \mathbb{H} \rightarrow \mathbb{H}$ by π_0 and π_τ , $(\tilde{f}_\tau)_*(\Gamma_0) = \theta_{f_\tau}(\Gamma_0) = \Gamma_\tau$. So by Lemma 5.1 of [9], we know that f_τ is homotopic to f^μ . So h is homotopic to f_τ^{-1} . The normalization of \tilde{h} , i.e. $\tilde{h}^*(\Gamma_0) = \Gamma_\tau$, comes trivially from the fact that $\Gamma_\tau = (f^\mu)_*(\Gamma_0)$ and $\tilde{h} = (f^\mu)^{-1}$. The uniqueness of such τ and h follows from the uniqueness of f^μ .

Now let us talk about the continuous dependence of (τ, h) on μ . For a continuous family of C^1 metrics $g(t)$, after pulling back to \mathbb{H} by π_0 , $g(t) = \lambda(t)|dz + \mu(t)d\bar{z}|^2$, and is continuous w.r.t t in the C^1 class. We have $|\mu(t)| \leq k(\epsilon) < 1$, and $\mu(t)$ continuous w.r.t. t in the C^1 class by Lemma 3.4. Let $f(t) = f^{\mu(t)}$ and $\tilde{h}(t) = (f(t))^{-1}$ as above.

Firstly, let us talk about the continuity of $\tau(t)$ w.r.t parameter t . Now the corresponding normalized Fuchsian model $\Gamma_{\tau(t)}$ is given by $(f^{\mu(t)})_*(\Gamma_0)$. Suppose the normalized generators for Γ_0 as in Section 2.5 of [9] are $\{\alpha_i^0, \beta_i^0\}_{i=1}^g$, where α_g^0 has attractive fixed point at 1 and β_g^0 has repelling and attractive fixed point at 0 and ∞ respectively. Then clearly $\{\theta_{f^{\mu(t)}}(\alpha_i^0), \theta_{f^{\mu(t)}}(\beta_i^0)\}_{i=1}^g$ form the normalized generators for $\Gamma_{\tau(t)}$. Now

$$\theta_{f^{\mu(t)}}(\gamma) = f^{\mu(t)} \circ \gamma \circ (f^{\mu(t)})^{-1} = f^{\mu(t)} \circ \gamma \circ \tilde{h}(t). \quad (19)$$

Now by Lemma 3.1.1 and Lemma 3.1.2, $f^{\mu(t)}$ and $\tilde{h}(t)$ are continuous w.r.t parameter t in C^0 class when acting on compact subsets of \mathbb{C} . So for fixed $\gamma \in \Gamma_0$, $\theta_{f^{\mu(t)}}(\gamma)$ is continuous w.r.t the parameter t , which means the coefficients of the linear fractional transformation corresponding to $\theta_{f^{\mu(t)}}(\gamma)$ are continuous functions of t . So the coefficients for $\{\theta_{f^{\mu(t)}}(\alpha_i^0), \theta_{f^{\mu(t)}}(\beta_i^0)\}_{i=1}^g$ are continuous functions of t . Now using the topology of Fricke Space as in Section 2.5 and Lemma 5.10 and Lemma 5.13 in [9], the corresponding elements $\tau(t) \in \mathcal{T}_g$ are continuous w.r.t the parameter t in the natural topology of \mathcal{T}_g .

Next, let us show the continuity of $h(t)$. Now lift up to $\tilde{h}(t) : \mathbb{H}_{dwd\bar{w}} \rightarrow \mathbb{H}_{|dz + \mu(t)d\bar{z}|^2}$, then $\tilde{h}(t) = (f^{\mu(t)})^{-1}$ is $\mu(t)$ quasi-linear quasi-conformal map as in Section 3.1.2. So by Lemma 3.3, we have the local $C^0 \cap W^{1,2}(\mathbb{H}, \mathbb{H})$ continuity of $\tilde{h}(t)$ w.r.t t , since $\mu(t)$ is continuous in C^1 w.r.t parameter t . It directly implies the continuity of $h(t) : \Sigma_{\tau(t)} \rightarrow \Sigma_0$ in the sense of Section 2.2, i.e. when restricting to compact subsets K of \mathbb{H} , the lifted up mapping $\tilde{h}(t) \in C^0([0, 1], C^0 \cap W^{1,2}(K, N))$.

□

3.3 Construction of the conformal reparametrization

As above, we consider $\tilde{g}_n(t) = \tilde{\gamma}_n(t)^*h$, which is continuous w.r.t t in the C^1 class by Lemma 3.1. Since $\tilde{g}_n(t)$ may be degenerate, let $g_n(t) = \tilde{g}_n(t) + \delta_n g_0$, where g_0 is the Poincaré metric of Σ_0 , and δ_n arbitrarily small. Then $g_n(t)$ uniquely determines $\tau_n(t) \subset \mathcal{T}_g$ and conformal diffeomorphisms $h_n(t)$ by Proposition 3.1. We have the following result similar to Theorem 3.1 of [15].

Theorem 3.1 *Using the above notations, we have reparametrizations $(\gamma_n(t), \tau_n(t)) \in \tilde{\Omega}$ for $\tilde{\gamma}_n(t)$, i.e. $\gamma_n(t) = \tilde{\gamma}_n(h_n(t), t)$, such that $\gamma_n(t) \in [\tilde{\gamma}_n]$. And*

$$E(\gamma_n(t), \tau_n(t)) - \text{Area}(\gamma_n(t)) \rightarrow 0, \quad (20)$$

for some sequence $\delta_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof: We know that $h_n(t) : \Sigma_{\tau_n(t)} \rightarrow (\Sigma, g_n(t))$ are conformal diffeomorphisms. Let $\gamma_n(t) = \tilde{\gamma}_n(h_n(t), t) : \Sigma_{\tau_n(t)} \rightarrow N$ be the composition with the almost conformal parametrization. To show that $\gamma_n(t)$ is a path in Ω , we only need to show the continuity. The continuity of $t \rightarrow \gamma_n(t)$ from $[0, 1]$ to $C^0 \cap W^{1,2}(\Sigma_{\tau_n(t)}, N)$ follows from the continuity of $t \rightarrow \tilde{\gamma}_n(t)$ in C^2 by Lemma 3.1, and that of $t \rightarrow h_n(t)$ in $C^0 \cap W^{1,2}(\Sigma_{\tau(t)}, \Sigma_0)$ by Proposition 3.1. Moreover $\gamma_n(t)$ is homotopic to $\tilde{\gamma}_n$. From our discussion of homotopy equivalence of mappings defined on different domains in Section 2.2, we view $\gamma_n(t)$ as mappings defined on Σ_0 by composing with $f_{\tau_n(t)} : \Sigma_0 \rightarrow \Sigma_{\tau_n(t)}$ and compare it to $\tilde{\gamma}_n(t)$. Since $h_n(t)$ are homotopic equivalent to $f_{\tau_n(t)}^{-1}$ by Proposition 3.1, $h_n(t) \circ f_{\tau_n(t)}$ is homotopic equivalent to identity map of Σ_0 . While γ_n are composition of $\tilde{\gamma}_n$ with $h_n(t)$, $\gamma_n \circ f_{\tau_n}$ is homotopic equivalent to $\tilde{\gamma}_n$, hence $\gamma_n \sim \tilde{\gamma}_n$.

We can get estimates as in Appendix D of [5] and the proof of Theorem 3.1 of [15]:

$$\begin{aligned} E(\gamma_n(t), \tau_n(t)) &= E(h_n(t) : T_{\tau_n(t)}^2 \rightarrow (\Sigma_0, \tilde{g}_n(t))) \leq E(h_n(t) : \Sigma_{\tau_n(t)} \rightarrow (\Sigma_0, g_n(t))) \\ &= \text{Area}(h_n(t) : \Sigma_{\tau_n(t)} \rightarrow (\Sigma_0, g_n(t))) \\ &= \text{Area}(\Sigma_0, g_n(t)) = \int_{\Sigma_0} [\det(g_n(t))]^{\frac{1}{2}} d\text{vol}_0 \\ &= \int_{\Sigma_0} [\det(\tilde{g}_n(t)) + \delta_n \text{Tr}_{g_0} \tilde{g}_n(t) + C(\tilde{g}_n(t)) \delta_n^2]^{\frac{1}{2}} d\text{vol}_0 \\ &\leq \text{Area}(\Sigma_0, \tilde{g}_n(t)) + C(\tilde{g}_n(t)) \sqrt{\delta_n} \\ &= \text{Area}(\gamma_n(t) : \Sigma_0 \rightarrow N) + C(\tilde{\gamma}_n) \sqrt{\delta_n}. \end{aligned} \quad (21)$$

The first and last equality follow from the definition of energy and area integral, and the second inequality is due to the fact $\tilde{g}_n(t) \leq g_n(t)$. Hence we have equation 20, if we choose $\delta_n \rightarrow 0$ depending only on $\tilde{\gamma}_n$.

□

Remark 3.2 *By argument similar to Proposition 1.5 in [5] and Remark 3.2 in [15], the above theorem implies that $\mathcal{W} = \mathcal{W}_E$.*

4 Compactification for mappings

For each $(\gamma_n(t), \tau_n(t))$ gotten above, $\tau_n(t)$ corresponds to a normalized Fuchsian model $(\Sigma_{\tau_n(t)}, \Gamma_{\tau_n(t)})$. We can also view $\gamma_n(t)$ as been lifted up to \mathbb{H} by $\pi_{\tau_n(t)} : \mathbb{H} \rightarrow \Sigma_{\tau_n(t)}$. Denote the lifted mappings again by $\gamma_n(t)$, then $\gamma_n(t)$ can be viewed as defined on the same domain \mathbb{H} , i.e. $\gamma_n(t) : \mathbb{H} \rightarrow N$, but invariant under different Fuchsian groups $\Gamma_{\tau_n(t)}$ action, i.e. $\forall \gamma \in \Gamma_{\tau_n(t)}, \gamma_n(t) \circ \gamma = \gamma_n(t)$. We can apply similar perturbation procedure to the lifted mappings as in [5][15].

Before doing such perturbations, let us firstly talk about collections of disjoint balls on Σ_τ . Here we use $\mathcal{B} = \cup_{i=1}^n B_i$ to denote a finite collection of disjoint geodesic balls on Σ_τ , with the radii of balls less than the injective radius r_{Σ_τ} of Σ_τ . Taking a ball $B \in \mathcal{B}$ with radius r_B , we would like to talk about a sub-geodesic ball with the same center but with the radius only a ratio $\mu < 1$ of r_B , which we denote by μB . Such a geodesic ball B with hyperbolic metric of curvature -1 can always pulled back to the Poincaré disk $(D, ds_{-1}^2 = \frac{|dx|^2}{1-|x|^2})$, such that the center of B goes to the center of D . Then B can be viewed as a disk $B(0, r_B^0) \subset D$ with hyperbolic metric ds_{-1}^2 , where r_B^0 is the Euclidean radius of the image of B and $r_B = \int_0^{r_B^0} \frac{1}{\sqrt{1-t^2}} dt = \sin^{-1}(r_B^0)$. The hyperbolic metric is now conformal and uniformly equivalent to the Euclidean metric $ds_0^2 = |dx|^2$ on B . Here *uniformly equivalent* means $C^{-1}ds_0^2 \leq ds_{-1}^2 \leq Cds_0^2$ for some constant $C > 1$. There exists a small number:

$$r_0 = \sin^{-1}\left(\frac{1}{2}\right), \quad (22)$$

such that if we restrict the radius r_B of B to be less than r_0 , we can choose the constant $C = \frac{4}{3}$. Then if we consider $\frac{1}{4}B$, then we know that in the Euclidean metric ds_0^2 , the radius of $\frac{1}{4}B$ is less than $\frac{1}{2}r_B^0$, i.e. $\frac{1}{4}B \subset B(0, \frac{1}{2}r_B^0)$. Later on, we will always assume that geodesic balls have their radii bounded from above by r_0 .

Lemma 4.1 *Let $[\beta]$ and \mathcal{W}_E be as defined in definition 2.2. For any $(\gamma(t), \tau(t)) \in [\beta] \subset \tilde{\Omega}$ with $\max_{t \in [0,1]} E(\gamma(t), \tau(t)) - \mathcal{W}_E \ll 1$, if $(\gamma(t), \tau(t))$ is not harmonic unless $\gamma(t)$ is a constant map, we can perturb $\gamma(t)$ to $\rho(t)$, such that $\rho(t) \in [\gamma]$ and $E(\rho(t), \tau(t)) \leq E(\gamma(t), \tau(t))$. Moreover for any t such that $E(\gamma(t), \tau(t)) \geq \frac{1}{2}\mathcal{W}_E$, $\rho(t)$ satisfy:*

(*) *For any finite collection of disjoint balls $\cup_i B_i$ on Σ_{τ_t} with geodesic radius of each ball B_i bounded above by the injective radius $r_{\Sigma_{\tau_t}}$ and r_0 , such that $E(\rho(t), \cup_i B_i) \leq \epsilon_0$, let v be the energy minimizing harmonic map with the same boundary value as $\rho(t)$ on $\frac{1}{64} \cup_i B_i$, then we have:*

$$\int_{\frac{1}{64} \cup_i B_i} |\nabla \rho(t) - \nabla v|^2 \leq \Psi \left(E(\gamma(t), \tau(t)) - E(\rho(t), \tau(t)) \right). \quad (23)$$

Here ϵ_0 is some small constant, and Ψ is a positive continuous function with $\Psi(0) = 0$.

Remark 4.1 *We will mainly use the idea in the proof of Theorem 2.1 of [5] and the proof of Lemma 4.1 of [15]. As talked in the remarks following Lemma 4.1 of [15], we would need to show the continuity of local harmonic replacement and comparison of energy decrease of successive harmonic replacements. The continuity of harmonic replacement is a conformal invariant property, which we can handle by pulling every ball we care to the center of the Poincaré disk as above. For the comparison of the energy decrease, it turns out that what we really need to care is the analysis on a single ball. So we could do that by pulling the chosen ball to the center of the Poincaré disk again, without caring about the image of the other balls.*

In the following three subsections, we will list the results about analysis of harmonic replacements on disks. Then we will give a result of comparison of harmonic replacements, where we will show a similar result to Lemma 3.11 of [5] and Lemma 4.2 of [15] by showing a proof with minor differences to the previous ones. At the end, we will give the deformation map $\gamma \rightarrow \rho$ by explicit constructions.

4.1 Results about harmonic replacements on disks

Here we summarize some known results of harmonic replacements on disks. Let B_1 be the unit disk in \mathbb{R}^2 , and N the ambient manifold.

Theorem 4.1 (Theorem 3.1 of [5]) *There exists a small constant ϵ_1 (depending only on N) such that for all maps $u, v \in W^{1,2}(B_1, N)$, if v is weakly harmonic with the same boundary value as u , and v has energy less than ϵ_1 , then we have:*

$$\int_{B_1} |\nabla_0 u|^2 - \int_{B_1} |\nabla_0 v|^2 \geq \frac{1}{2} \int_{B_1} |\nabla_0 u - \nabla_0 v|^2. \quad (24)$$

Here we use ∇_0 to denote the flat connection of B_1 .

Remark 4.2 *Although this theorem is formulated when we use the standard metric $ds_0^2 = dx^2 + dy^2$ on B_1 , we can still have inequality 24, if we take another metric ds^2 on B_1 which is conformal to ds_0^2 , since both sides of inequality 24 are conformal invariant. So if we take the standard hyperbolic metric ds_{-1}^2 on a small ball as talked in the beginning of Section 4, inequality 24 is still true only by changing the flat connection to the connection ∇ of ds_{-1}^2 .*

Remark 4.3 *As talked in Section 4.2 of [15], we can use the energy gap to control the $W^{1,2}$ -norm difference between a mapping defined on the unit disk with its corresponding energy minimizing harmonic mapping with the same boundary data. This theorem also implies the uniqueness of energy minimizing harmonic maps with energy less than ϵ_1 and fixed boundary values (Corollary 3.3 of [5]).*

Based on this theorem, we have the following result which shows that perturbing mappings locally to energy minimizing harmonic mappings is a continuous functional. This is a combination of Corollary 4.1 and 4.2 of [15], so here we omit the proof.

Corollary 4.1 (Corollary 3.4 of [5], Corollary 4.1 and 4.2 of [15]) *Let ϵ_1 be given in the previous theorem. Suppose $u \in C^0(\overline{B}_1) \cap W^{1,2}(B_1)$ with energy $E(u) \leq \epsilon_1$, then there exists a unique energy minimizing harmonic map $v \in C^0(\overline{B}_1) \cap W^{1,2}(B_1)$ with the same boundary value as u . Set $\mathcal{M} = \{u \in C^0(\overline{B}_1) \cap W^{1,2}(B_1) : E(u) \leq \epsilon_1\}$. If we denote v by $H(u)$, the map $H : \mathcal{M} \rightarrow \mathcal{M}$ is continuous w.r.t the norm on $C^0(\overline{B}_1) \cap W^{1,2}(B_1)$. Here the norm is the sum of $C^0(\overline{B}_1)$ -norm and $W^{1,2}(B_1)$ -norm.*

Suppose u_i, u are defined on a ball $B_{1+\epsilon}$ with energy less than ϵ_1 . Suppose $u_i \rightarrow u$ in $C^0(\overline{B}_{1+\epsilon}) \cap W^{1,2}(B_{1+\epsilon})$. Choose a sequence $r_i \rightarrow 1$, and let w_i, w be the mappings which coincide with u_i, u outside $r_i B_1$ and B_1 and are energy minimizing inside $r_i B_1$ and B_1 respectively. Then $w_i \rightarrow w$ in $C^0(\overline{B}_{1+\epsilon}) \cap W^{1,2}(B_{1+\epsilon})$.

Remark 4.4 *If we use geodesic ball B_r of geodesic radius $r \leq r_0$ on Σ_g with Poincaré metric, all the results of the above lemma hold. This is because that the Poincaré metric ds_{-1}^2 is conformal and uniformly equivalent to the flat metric ds_0^2 , so harmonic maps w.r.t. ds_0^2 are also harmonic w.r.t. ds_{-1}^2 , and C^0 and $W^{1,2}$ norms of a fixed map w.r.t. ds_{-1}^2 are uniformly equivalent to those w.r.t. ds_0^2 .*

4.2 Comparison results of successive harmonic replacements

Now we will give a comparison result for successive harmonic replacements by adapting Lemma 3.11 in [5] and Lemma 4.2 in [15]. Fix a mapping $u \in W^{1,2}(\Sigma_g, N)$. We still denote \mathcal{B} as a finite collection of disjoint geodesic balls on Σ_g as above. Given $\mu \in [0, 1]$, denote $\mu\mathcal{B}$ to be the collection of geodesic balls with the same centers as \mathcal{B} , but with geodesic radii μ times those corresponding ones of \mathcal{B} . Suppose that u has small energy on a collection \mathcal{B} . We denote $H(u, \mathcal{B})$ to be the mapping which coincides with u outside \mathcal{B} , but are the energy minimizing ones inside \mathcal{B} with the same boundary values as u on $\partial\mathcal{B}$. We call H the harmonic replacement in the following. If $\mathcal{B}_1, \mathcal{B}_2$ are two such collections, we denote $H(u, \mathcal{B}_1, \mathcal{B}_2)$ to be $H(H(u, \mathcal{B}_1), \mathcal{B}_2)$. We have the following energy comparison results for u , $H(u, \mathcal{B}_1)$ and $H(u, \mathcal{B}_1, \mathcal{B}_2)$.

Lemma 4.2 *Fix a Riemann surface Σ_g with Poincaré metric, and a mapping $u \in C^0 \cap W^{1,2}(\Sigma_g, N)$. Let $\mathcal{B}_1, \mathcal{B}_2$ be two finite collections of disjoint geodesic balls on Σ_g with radii less than the injective radius r_{Σ_g} and r_0 as (22). If $E(u, \mathcal{B}_i) \leq \frac{1}{3}\epsilon_1$ for $i = 1, 2$, with ϵ_1 given in Theorem 4.1, then there exists a constant k depending on N , such that:*

$$E(u) - E[H(u, \mathcal{B}_1, \mathcal{B}_2)] \geq k \left(E(u) - E[H(u, \frac{1}{4}\mathcal{B}_2)] \right)^2, \quad (25)$$

and for any $\mu \in [\frac{1}{64}, \frac{1}{4}]$,

$$\frac{1}{k} \left(E(u) - E[H(u, \mathcal{B}_1)] \right)^{\frac{1}{2}} + E(u) - E[H(u, 4\mu\mathcal{B}_2)] \geq E[H(u, \mathcal{B}_1)] - E[H(u, \mu\mathcal{B}_2)]. \quad (26)$$

Remark 4.5 *The proof is very similar to that of Lemma 4.2 of [15]. We will use the Euclidean metric which is conformal to the hyperbolic metric on each of the geodesic balls we are considering. Since the inequalities 25 and 26 are all conformal invariant, the proof in the Euclidean metrics implies that in hyperbolic metrics. By the energy minimizing properties, we can easily get the following inequality:*

$$E(u) - E[H(u, \mathcal{B}_1, \mathcal{B}_2)] \geq E(u) - E[H(u, \frac{1}{4}\mathcal{B}_1)]. \quad (27)$$

This is because that $E[H(u, \mathcal{B}_1, \mathcal{B}_2)] \leq E[H(u, \mathcal{B}_1)] \leq E[H(u, \frac{1}{4}\mathcal{B}_1)]$. Combining the above inequalities, we get the comparison for energy of any two successive harmonic replacements by appropriately shrinking the radii.

We need the following lemma to construct comparison maps. This is a scaling invariant version.

Lemma 4.3 (Lemma 3.14 of [5]) *There exists a δ and a large constant C depending on N , such that for any $f, g \in C^0 \cap W^{1,2}(\partial B_R, N)$, if f, g are equal at some point on ∂B_R , and:*

$$R \int_{\partial B_R} |f' - g'|^2 \leq \delta^2, \quad (28)$$

we can find some $\rho \in (0, \frac{1}{2}R]$, and a mapping $w \in C^0 \cap W^{1,2}(B_R \setminus B_{R-\rho}, N)$ with $w|_{B_R} = f$, $w|_{B_{R-\rho}} = g$, which satisfies estimates:

$$\int_{B_R \setminus B_{R-\rho}} |\nabla w|^2 \leq C \left(R \int_{\partial B_R} |f'|^2 + |g'|^2 \right)^{\frac{1}{2}} \left(R \int_{\partial B_R} |f' - g'|^2 \right)^{\frac{1}{2}}. \quad (29)$$

Proof: (of Lemma 4.2) Here we will adapt the proof of Lemma 4.2 in [15]. Since we assume that $E(u, \mathcal{B}_i) \leq \frac{1}{3}\epsilon_1$, we know that u and $H(u, \mathcal{B}_1)$ have energy less than $\frac{2}{3}\epsilon_1$ on $\mathcal{B}_1 \cup \mathcal{B}_2$, so we can use energy gaps to control $W^{1,2}$ norms difference by Theorem 4.1. Denote balls in \mathcal{B}_1 by B_α^1 , and balls in \mathcal{B}_2 by B_j^2 . We prove the two inequalities separately.

1° Inequality 25: We divide the second collection \mathcal{B}_2 into two sub-collections $\mathcal{B}_2 = \mathcal{B}_{2+} \cup \mathcal{B}_{2-}$, where $\mathcal{B}_{2+} = \{B_j^2 : \frac{1}{4}B_j^2 \subset B_\alpha^1 \text{ or } \frac{1}{4}B_j^2 \cap \mathcal{B}_1 = \emptyset \text{ for some } B_\alpha^1 \in \mathcal{B}_1\}$ and $\mathcal{B}_{2-} = \mathcal{B}_2 \setminus \mathcal{B}_{2+}$, and deal with them separately.

For collection \mathcal{B}_{2+} , we separate it into another two sub-collections $\{\frac{1}{4}B_j^2 \cap \mathcal{B}_1 = \emptyset\}$ and $\{\frac{1}{4}B_j^2 \subset B_\alpha^1\}$. For balls $\frac{1}{4}B_j^2 \cap \mathcal{B}_1 = \emptyset$, we can use the energy minimizing property of small energy harmonic maps as in remark 4.3, and similar arguments as inequality (18) and (19) in [15] to get,

$$\sum_{\{\frac{1}{4}B_j^2 \cap \mathcal{B}_1 = \emptyset\}} (E(u) - E[H(u, \frac{1}{4}B_j^2)]) \leq E(u) - E[H(u, \mathcal{B}_1, \cup_{\frac{1}{4}B_j^2 \cap \mathcal{B}_1 = \emptyset} B_j^2)]. \quad (30)$$

For balls $\frac{1}{4}B_j^2 \subset B_\alpha^1$, $H(u, \mathcal{B}_1, \frac{1}{4}B_j^2) = H(u, \mathcal{B}_1)$. We denote $u_1 = H(u, \mathcal{B}_1)$. Using energy minimizing property of small energy harmonic maps again, and similar arguments

as inequality (20) and (21) of [15], we have,

$$\begin{aligned} \int_{\frac{1}{4}B_j^2 \cup B_\alpha^1}^{B_j^2} |\nabla u|^2 - |\nabla H(u, \frac{1}{4}B_j^2)|^2 &\leq \int_{\frac{1}{4}B_j^2 \cup B_\alpha^1}^{B_j^2} |\nabla u|^2 - |\nabla H(u, \mathcal{B}_1, B_j^2)|^2 \\ &\leq \int |\nabla u|^2 - |\nabla u_1|^2 + \int_{\frac{1}{4}B_j^2 \cup B_\alpha^1}^{B_j^2} |\nabla u_1|^2 - |\nabla H(u, \mathcal{B}_1, B_j^2)|^2 \end{aligned} \quad (31)$$

The second " \leq " of the above is gotten by adding a term $\int_{\frac{1}{4}B_j^2 \cup B_\alpha^1}^{B_j^2} |\nabla u_1|^2$ and subtracting a same term after the first " \leq ". For the first term, using Theorem 4.1 and the following Remark 4.2, we have that $\int |\nabla u|^2 - |\nabla u_1|^2 \leq \int |\nabla u - \nabla u_1|^2 \leq 4(E(u) - E(u_1))$. The second term is bounded from above by $E(u_1) - E[H(u_1, \frac{1}{4}B_j^2 \cup B_\alpha^1)] \leq E(u) - E[H(u, \mathcal{B}_1, \frac{1}{4}B_j^2 \cup B_\alpha^1)]$. So combining the above estimates together, we get inequality,

$$E(u) - E[H(u, \frac{1}{4}\mathcal{B}_{2+})] \leq C(E(u) - E[H(u, \mathcal{B}_1, \mathcal{B}_{2+})]). \quad (32)$$

Now let us consider the sub-collection \mathcal{B}_{2-} . Here we deal with balls individually. Fix a $B_j^2 \in \mathcal{B}_{2-}$, then $\frac{1}{4}B_j^2 \cap B_\alpha^1 \neq \emptyset$ for some $B_\alpha^1 \in \mathcal{B}_1$, but $\frac{1}{4}B_j^2$ does not belong to any $B_\alpha^1 \in \mathcal{B}_1$. Using discussions about small geodesic balls in the beginning of Section 4, we can identify this B_j^2 with a sub-disk centered at the origin of the Poincaré disk, and model it by $(B(0, r_B^0), \frac{ds_0^2}{1-|x|^2})$. Simply denote it by $B_{r_B^0}$, and denote $u_1 = H(u, \mathcal{B}_1)$ as above. Lower subindex here is used to denote the radius of that ball w.r.t. ds_0^2 . Now let us construct an auxiliary comparison map. Using basic measure theory, there exists a subset of $[\frac{3}{4}r_B^0, r_B^0]$ with measure $\frac{1}{36}r_B^0$, such that for any r in this subset, we have,

$$\int_{\partial B_r} |\nabla_0 u_1 - \nabla_0 u|^2 \leq \frac{9}{r_B^0} \int_{\frac{3}{4}r_B^0}^{r_B^0} \int_{\partial B_s} |\nabla_0 u_1 - \nabla_0 u|^2 \leq \frac{9}{r} \int_{B_{r_B^0}} |\nabla_0 u_1 - \nabla_0 u|^2, \quad (33)$$

$$\int_{\partial B_r} |\nabla_0 u_1|^2 + |\nabla_0 u|^2 \leq \frac{9}{r_B^0} \int_{\frac{3}{4}r_B^0}^{r_B^0} \int_{\partial B_s} |\nabla_0 u_1|^2 + |\nabla_0 u|^2 \leq \frac{9}{r} \int_{B_{r_B^0}} |\nabla_0 u_1|^2 + |\nabla_0 u|^2, \quad (34)$$

where ∇_0 is the connection of ds_0^2 . By choosing ϵ_1 small enough, we can make $r \int_{\partial B_r} |\nabla_0 u_1|^2 + |\nabla_0 u|^2 \leq \delta^2$ and $r \int_{\partial B_r} |\nabla_0 u_1 - \nabla_0 u|^2 \leq \delta^2$ with δ as in the above Lemma 4.3. Since $\frac{1}{4}B_{r_B^0} \subset B_{\frac{1}{2}r_B^0}$ as discussed in the beginning of Section 4, and that $B_{r_B^0} \in \mathcal{B}_{2-}$, $B_{\frac{1}{2}r_B^0}$ and hence B_r must intersect a ball in \mathcal{B}_1 but is not contained in any ball of \mathcal{B}_1 , so u

and u_1 must coincide at least one point on ∂B_r . So by Lemma 4.3, $\exists \rho \in (0, \frac{1}{2}r]$ and $\exists w \in C^0 \cap W^{1,2}(B_r \setminus B_{r-\rho})$ with $w|_{\partial B_r} = u_1|_{\partial B_r}$, $w|_{\partial B_{r-\rho}} = u|_{\partial B_r}$, and:

$$\begin{aligned} \int_{B_r \setminus B_{r-\rho}} |\nabla_0 w|^2 &\leq C \left(r \int_{\partial B_r} |\nabla_0 u_1 - \nabla_0 u|^2 \right)^{\frac{1}{2}} \left(r \int_{\partial B_r} |\nabla_0 u_1|^2 + |\nabla_0 u|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{B_{r_B}^0} |\nabla_0 u_1 - \nabla_0 u|^2 \right)^{\frac{1}{2}} \left(\int_{B_{r_B}^0} |\nabla_0 u_1|^2 + |\nabla_0 u|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (35)$$

Now construct comparison map v on $B_{r_B}^0$ such that:

$$v = \begin{cases} u_1 & \text{on } B_{r_B}^0 \setminus B_r \\ w & \text{on } B_r \setminus B_{r-\rho} \\ H(u, B_r)(\frac{r}{r-\rho}x) & \text{on } B_{r-\rho} \end{cases}.$$

In the last equation, we do a scaling w.r.t. the flat coordinates. Now $E[H(u_1, B_{r_B}^0)] \leq E(v)$ on $B_{r_B}^0$, since $H(u_1, B_{r_B}^0)$ is the energy minimizing harmonic maps among maps with the same boundary values. So:

$$\begin{aligned} \int_{B_{r_B}^0} |\nabla_0 H(u_1, B_{r_B}^0)|^2 &\leq \int_{B_{r_B}^0} |\nabla_0 v|^2 \\ &= \int_{B_{r_B}^0 \setminus B_r} |\nabla_0 u_1|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla_0 w|^2 + \int_{B_{r-\rho}} |\nabla_0 H(u, B_r)(\frac{r}{r-\rho} \cdot)|^2 \\ &= \int_{B_{r_B}^0 \setminus B_r} |\nabla_0 u_1|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla_0 w|^2 + \int_{B_r} |\nabla_0 H(u, B_r)|^2. \end{aligned} \quad (36)$$

Now since $\frac{1}{4}B_{r_B}^0 \subset B_{\frac{1}{2}r_B}^0 \subset B_r$, we have:

$$\begin{aligned} \int_{\frac{1}{4}B_{r_B}^0} |\nabla_0 u|^2 - \int_{\frac{1}{4}B_{r_B}^0} |\nabla_0 H(u, \frac{1}{4}B_{r_B}^0)|^2 &\leq \int_{B_r} |\nabla_0 u|^2 - \int_{B_r} |\nabla_0 H(u, B_r)|^2 \\ &\leq \int_{B_r} |\nabla_0 u|^2 - \int_{B_{r_B}^0} |\nabla_0 H(u_1, B_{r_B}^0)|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla_0 w|^2 + \int_{B_{r_B}^0 \setminus B_r} |\nabla_0 u_1|^2 \\ &\leq \int_{B_{r_B}^0} |\nabla_0 u_1|^2 - \int_{B_{r_B}^0} |\nabla_0 H(u_1, B_{r_B}^0)|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla_0 w|^2 + \int_{B_r} |\nabla_0 u|^2 - \int_{B_r} |\nabla_0 u_1|^2. \end{aligned} \quad (37)$$

Now we can use the conformal invariance for energy integral to change all the flat connection ∇_0 and flat metric ds_0^2 to hyperbolic connection ∇ and hyperbolic metric ds_{-1}^2 . Summing the above inequality on all balls in \mathcal{B}_{2-} , and using Theorem 4.1 and the

following Remark 4.2 together with inequality 35, we can get the following inequality by similar arguments as those in inequalities (29) and (30) in [15]:

$$E(u) - E[H(u, \frac{1}{4}\mathcal{B}_{2-})] \leq C'(E(u) - E[H(u, \mathcal{B}_1, \mathcal{B}_2)])^{\frac{1}{2}}. \quad (38)$$

Combing inequalities on \mathcal{B}_{2+} and \mathcal{B}_{2-} , we get the inequality 25.

2° Inequality 26: We divide \mathcal{B}_2 into two disjoint sub-collections \mathcal{B}_{2+} and \mathcal{B}_{2-} , with $\mathcal{B}_{2+} = \{B_j^2 : \mu B_j^2 \subset B_\alpha^1 \text{ or } \mu B_j^2 \cap \mathcal{B}_1 = \emptyset\}$. For collection \mathcal{B}_{2+} , similar method also gives:

$$E[H(u, \mathcal{B}_1)] - E[H(u, \mathcal{B}_1, \mu\mathcal{B}_{2+})] \leq E(u) - E[H(u, 4\mu\mathcal{B}_{2+})]. \quad (39)$$

For subcollection \mathcal{B}_{2-} , we use similar proof as above. Here we identify $4\mu B_j^2$ with a sub-disk centered at the origin of the Poincaré disk again, and get an isometric representation $(B_{r_B^0}, ds_{-1}^2)$. In the construction of w , we change the role of u and u_1 . Let the comparison map be,

$$v = \begin{cases} u & \text{on } B_{r_B^0} \setminus B_r \\ w & \text{on } B_r \setminus B_{r-\rho} \\ H(u_1, B_r)(\frac{r}{r-\rho} x) & \text{on } B_{r-\rho} \end{cases}.$$

We have $\int_{B_{r_B^0}} |\nabla_0 H(u, B_{r_B^0})|^2 \leq \int_{B_{r_B^0}} |\nabla_0 v|^2$ by the energy minimizing property. Since we have $\mu B_j^2 = \frac{1}{4} B_{r_B^0} \subset B_{\frac{1}{2}r_B^0}$, by argument similar to inequalities (34)(35) and (36) in [15], we get,

$$E(u_1) - E[H(u_1, \mu\mathcal{B}_{2-})] \leq E(u) - E[H(u, 4\mu\mathcal{B}_{2-})] + C(E(u) - E(u_1))^{\frac{1}{2}}. \quad (40)$$

Combining results on \mathcal{B}_{2+} and \mathcal{B}_{2-} , we get inequality 26.

□

4.3 Construction of deformation maps

Let us talk about harmonic replacements on paths $(\gamma(t), \tau(t)) \in \tilde{\Omega}$ now. The normalized Fuchsian models of $\tau(t)$ are given by $(\Sigma_{\tau(t)}, \Gamma_{\tau(t)})$, and denote the injective radius of $\Sigma_{\tau(t)}$ by $r_{\tau(t)}$. Firstly, let us point out where to do harmonic replacements. Fix a time parameter $t \in (0, 1)$. Suppose B is a ball on $\Sigma_{\tau(t)}$, with radius $r_B < r_{\tau(t)}$. As discussed in the beginning of this section, we can view $\gamma(t)$ as been defined on the upper half plan \mathbb{H} by lifting up using $\pi_{\tau(t)} : \mathbb{H} \rightarrow \Sigma_{\tau(t)}$. Since $\{\tau(t)\}$ is a compact set in

\mathcal{T}_g , we can always pick one connected component of the pre-images $\pi_{\tau(t)}^{-1}(B)$ inside a fix compact subset $K \subset \mathbb{H}$. Denote that connected component still by B , then obviously it has radius r_B w.r.t the hyperbolic metric ds_{-1}^2 of \mathbb{H} . Moreover B is a standard ball in \mathbb{H} w.r.t. the flat metric ds_0^2 . By the continuity of $\tau(t)$, for parameter $|s - t| \ll 1$, we have that the injective radius $|r_{\tau(s)} - r_{\tau(t)}| \ll 1$. So $r_B < r_{\tau(s)}$, hence the image of this ball B under $\pi_{\tau(s)} : \mathbb{H} \rightarrow \Sigma_{\tau(s)}$ is also a geodesic ball with radius less than the injective radius $r_{\tau(s)}$ of Σ_s . Denoting the image by B again, we want to do harmonic replacement simultaneously on $B \subset \Sigma_s$ for $|s - t| \ll 1$.

When $|s - t| \ll 1$, let us pick up a continuous cutoff function $\mu(s)$, such that $\mu(t) = 1$, and $\mu(s) = 0$ for $|s - t| > \delta$ with $\delta > 0$ small enough. If we do harmonic replacements for $\gamma(s)$ on balls $\mu(s)B$, Corollary 4.1 and discussion in Remark 4.4 together with the definition of continuity of pathes directly imply that we get another continuous path in $\tilde{\Omega}$. Similarly, we can continuously shrink the radii on balls $\mu(s)B$ where we do harmonic replacements continuously to 0, so that the new path can be continuously deformed to the original one in $\tilde{\Omega}$, which implies that they lie in the same homotopy class by the definition of homotopy equivalence in Section 2.2.

The strategy to construct the deformation map is to do harmonic replacement firstly on a collection of disjoint geodesic balls where the energy decrease is almost maximal, and then use Lemma 4.2 to get estimate of form (23) for any other harmonic replacements on collection of balls with small energy. For $\sigma \in C^0 \cap W^{1,2}(\Sigma_\tau, N)$, $\epsilon \in (0, \epsilon_1]$, define the maximal possible energy decrease as,

$$e_{\epsilon, \sigma} = \sup_B \{E(\sigma, \tau) - E[H(\sigma, \frac{1}{4}\mathcal{B}), \tau]\}, \quad (41)$$

where \mathcal{B} are chosen as any finite collection of disjoint geodesic balls on Σ_τ with radii less than r_0 as in (22) and the injective radius of Σ_τ , satisfying: $E(\sigma, \mathcal{B}) \leq \epsilon$. When σ is not harmonic, we always have that $e_{\epsilon, \sigma} > 0$. Now for a path $(\sigma(t), \tau(t)) \in \tilde{\Omega}$, we have the following continuity property similar to Lemma 3.34 in [5] and Lemma 4.4 in [15].

Lemma 4.4 *$\forall t \in (0, 1)$, if $\sigma(t)$ is not harmonic, there exists a neighborhood $I^t \subset (0, 1)$ of t depending on t , ϵ and the path σ , such that $\forall s \in 2I^t$.*

$$e_{\frac{1}{2}\epsilon, \sigma(s)} \leq 2e_{\epsilon, \sigma(t)}. \quad (42)$$

Proof: Since $e_{\epsilon, \sigma(t)} > 0$, the continuity of $\sigma(s)$ implies that there exists a neighborhood \tilde{I}^t of t , such that $\forall s \in 2\tilde{I}^t$, and for any finite collection of balls $\mathcal{B} \subset K$, where

K is a fixed compact subset of \mathbb{H} ,

$$\frac{1}{2} \int_{\mathcal{B}} |\nabla \sigma(s) - \nabla \sigma(t)|^2 \leq \min\left\{\frac{1}{4}e_{\epsilon, \sigma(t)}, \frac{1}{2}\epsilon\right\}, \quad (43)$$

where we view $\sigma(s)$ as being lifted up to \mathbb{H} .

Fix $s \in 2\tilde{I}^t$. By definition 41, we can pick a finite collection of balls $\mathcal{B} \subset \Sigma_{\tau(s)}$, such that $E(\sigma(s), \mathcal{B}) \leq \frac{1}{2}\epsilon$ and $E(\sigma(s)) - E[H(\sigma(s), \frac{1}{4}\mathcal{B})] \geq \frac{3}{4}e_{\frac{1}{2}\epsilon, \sigma(s)}$. By taking the compact set $K \subset \mathbb{H}$ large enough, we can always find a connected pre-image in K for each ball in \mathcal{B} . Denote those connected pre-image balls by \mathcal{B} again. Then take the image of \mathcal{B} under $\pi_{\tau(t)} : \mathbb{H} \rightarrow \Sigma_{\tau(t)}$, we get another collection of geodesic balls on $\Sigma_{\tau(t)}$, which we still denote by \mathcal{B} . So $E(\sigma(t), \mathcal{B}) \leq E(\sigma(s), \mathcal{B}) + \frac{1}{2}\epsilon \leq \epsilon$ by (43), hence $E(\sigma(t)) - E[H(\sigma(t), \frac{1}{4}\mathcal{B})] \leq e_{\epsilon, \sigma(t)}$ by definition 41. So

$$\begin{aligned} & E(\sigma(s)) - E[H(\sigma(s), \frac{1}{4}\mathcal{B})] \\ & \leq |E(\sigma(s)) - E(\sigma(t))| + E(\sigma(t)) - E[H(\sigma(t), \frac{1}{4}\mathcal{B})] \\ & \quad + |E[H(\sigma(t), \frac{1}{4}\mathcal{B})] - E[H(\sigma(s), \frac{1}{4}\mathcal{B})]|. \end{aligned} \quad (44)$$

Using the continuity of harmonic replacement, i.e. Corollary 4.1, we can possibly shrink the neighborhood \tilde{I}^t to a smaller one I^t , such that $|E(\sigma(s)) - E(\sigma(t))| \leq \frac{1}{4}e_{\epsilon, \sigma(t)}$ and $|E[H(\sigma(t), \frac{1}{4}\mathcal{B})] - E[H(\sigma(s), \frac{1}{4}\mathcal{B})]| \leq \frac{1}{4}e_{\epsilon, \sigma(t)}$. Hence $E(\sigma(s)) - E[H(\sigma(s), \frac{1}{4}\mathcal{B})] \leq \frac{3}{2}e_{\epsilon, \sigma(t)}$, so $e_{\frac{1}{2}\epsilon, \sigma(s)} \leq 2e_{\epsilon, \sigma(t)}$.

□

Next, we will choose families of collections of disjoint geodesic balls corresponding to paths $(\gamma(t), \tau(t)) \in \tilde{\Omega}$.

Lemma 4.5 *There exist a covering $\{I^{t_j} : j = 1, \dots, m\}$ for parameter space $[0, 1]$, and m collection of disjoint geodesic balls $\mathcal{B}_j \subset \Sigma_{\tau(t_j)}$, $j = 1, \dots, m$ having radii less than r_0 in (22) and the injective radius $r_{\tau(t_j)}$, together with continuous functions $r_j : [0, 1] \rightarrow [0, 1]$, $j = 1, \dots, m$, satisfying:*

- 1°. *Each $r_j(t)$ is supported in I^{t_j} ;*
- 2°. *For a fixed t , at most two r_j are positive, and $E(\gamma(t), r_j(t)\mathcal{B}_j) \leq \frac{1}{3}\epsilon_1$;*
- 3°. *If $t \in [0, 1]$, such that $E(\gamma(t), \tau(t)) \geq \frac{1}{2}\mathcal{W}$, there exists a j , such that $E(\gamma(t)) - E[H(\gamma(t), \frac{1}{4}r_j(t)\mathcal{B}_j)] \geq \frac{1}{8}e_{\frac{1}{8}\epsilon_1, \gamma(t)}$.*

The proof use continuity of the paths and $e_{\epsilon, \gamma(t)}$ together with a covering argument for the parameter space $[0, 1]$. It is similar to that of Lemma 3.39 in [5] and Lemma 4.5 in [15], so we left it for readers.

Proof: (of Lemma 4.1) The perturbation from $\gamma(t)$ to $\rho(t)$ is done by successive harmonic replacements on the collection of balls given in Lemma 4.5. Denote $\gamma^0(t) = \gamma(t)$, and $\gamma^k(t) = H(\gamma^{k-1}(t), r_k(t)\mathcal{B}_k)$, for $k = 1, \dots, m$. Then $\rho(t) = \gamma^m(t)$. Here we can shrink the length of each interval I^{t_j} , such that the harmonic replacements from $\gamma(t)$ to $\rho(t)$ keep the continuity of the paths as discussed in the beginning of this section. The homotopy equivalent of $\rho \in [\gamma]$ is also a consequence of the discussions in the beginning of this section. Since harmonic replacements decrease energy, we have $E(\rho(t)) \leq E(\gamma(t))$.

Now the property (*) comes from similar argument as in the proof of Lemma 4.1 of [15] which originate from the proof of Theorem 3.1 of [5]. For t such that $E(\gamma(t), \tau(t)) \geq \frac{1}{2}\mathcal{W}$, we deform $\gamma(t)$ to $\rho(t)$ by at most two harmonic replacements, with the possible middle one denoted by $\gamma^k(t)$. Now we focus on the case of two replacements, and the other case is similar and much easier. For any collection \mathcal{B} with $E(\rho(t), \mathcal{B}) \leq \frac{1}{12}\epsilon_1$, we can assume that both $\gamma(t)$ and $\gamma^k(t)$ have energy less than $\frac{1}{8}\epsilon_1$ on \mathcal{B} , or inequality 23 is trivial. By property 3° of Lemma 4.5, at least one of the energy decrease from $\gamma(t)$ to $\rho(t)$ is bounded from below by $\frac{1}{8}e_{\frac{1}{8}\epsilon_1, \gamma(t)}$. so we have from either inequality 25 of Lemma 4.2 or inequality 27 that:

$$E(\gamma(t)) - E(\rho(t)) \geq k \left(\frac{1}{8} e_{\frac{1}{8}\epsilon_1, \gamma(t)} \right)^2. \quad (45)$$

Now using inequality 26 twice for $\mu = \frac{1}{64}, \frac{1}{16}$, we get:

$$\begin{aligned} & E(\rho(t)) - E[H(\rho(t), \frac{1}{64}\mathcal{B})] \\ & \leq E(\gamma^k(t)) - E[H(\gamma^k(t), \frac{1}{16}\mathcal{B})] + \frac{1}{k} \{ E(\gamma^k(t)) - E(\rho(t)) \}^{\frac{1}{2}} \\ & \leq E(\gamma(t)) - E[H(\gamma(t), \frac{1}{4}\mathcal{B})] + \frac{1}{k} \{ E(\gamma(t)) - E(\gamma^k(t)) \}^{\frac{1}{2}} \\ & \quad + \frac{1}{k} \{ E(\gamma(t)) - E(\rho(t)) \}^{\frac{1}{2}} \\ & \leq e_{\frac{1}{8}\epsilon_1, \gamma(t)} + C \{ E(\gamma(t)) - E(\rho(t)) \}^{\frac{1}{2}} \leq C \{ E(\gamma(t)) - E(\rho(t)) \}^{\frac{1}{2}}. \end{aligned} \quad (46)$$

By taking $\epsilon_0 = \frac{1}{12}\epsilon_1$ and Ψ a square root function, we can get inequality 23 by using Theorem 4.1 to change the left hand side to $W^{1,2}$ norm difference.

□

5 Convergence results

Here we talk about the convergence about our deformed sequences $\{\rho_n(t), \tau_n(t)\}_{n=1}^\infty$. In Lemma 4.1, we need our sequence $\{\gamma_n(t), \tau_n(t)\}_{n=1}^\infty$ to have no non-constant harmonic slices. We can achieve this by an argument similar to Remark 4.6 of [15]. In fact, we can modify the minimizing sequence $\{\tilde{\gamma}_n(t)\}_{n=1}^\infty$ such that $\tilde{\gamma}_n(t)$ are constant mappings on a small open neighborhood on Σ_0 , without changing the area too much. By Theorem 3.1, $\gamma_n(t)$ are gotten from $\tilde{\gamma}_n(t)$ by composing with diffeomorphisms $h_n(t)$, so $\gamma_n(t)$ are also constant mappings on some small neighborhood. By the unique continuation of harmonic maps (Corollary 2.6.1 of [10]), we know that for any parameter t , $\gamma_n(t)$ could not be harmonic mapping unless constant. So we can apply Lemma 4.1.

We would also like to preserve the almost conformal property given in Theorem 3.1 after the deformation given by Lemma 4.1. Although we could not make sure that $\rho_n(t)$ are still almost conformal for every parameter t after the deformation, we can prove similar results for the parameter t with $E(\rho_n(t_n), \tau_n(t_n))$ closed to the min-max critical value \mathcal{W} . The proof is almost the same as Lemma 5.1 of [15], so we omit the proof here. The result is as following.

Lemma 5.1 *Given a sequence of parameters $\{t_n\}_{n=1}^\infty$, such that $E(\rho_n(t_n), \tau_n(t_n)) \rightarrow \mathcal{W}$, then we have*

$$E(\rho_n(t_n), \tau_n(t_n)) - \text{Area}(\rho_n(t_n)) \rightarrow 0. \quad (47)$$

5.1 Degeneration of conformal structures

Let us talk about the **compactification of moduli space** \mathcal{M}_g . Here we mainly refer to Appendix B of [9] and Chapter IV of [8]¹². In fact, we will use hyperbolic metrics to represent elements in \mathcal{M}_g and its compactification. Firstly, let us talk about the representation of the moduli space \mathcal{M}_g and Teichmüller space \mathcal{T}_g by hyperbolic and complex structures. Fix a topological surface Σ_0 of genus $g \geq 2$. In fact, every metric on Σ_0 determines a complex structure j . There exists a hyperbolic metric h compatible with j . In fact, by uniformization theorem the covering projection $\pi : \mathbb{H} \rightarrow (\Sigma_0, j)$ is holomorphic, and the deck transformation group acts isomorphically w.r.t. the

¹²Section 4 of [16] also gives a nice summation in hyperbolic structures.

hyperbolic metric ds_{-1}^2 . So we can get a hyperbolic metric h on Σ_0 by pushing down ds_{-1}^2 , and this metric is compatible to j since ds_{-1}^2 is compatible to the standard complex structure on \mathbb{H} . Denote such a hyperbolic Riemann surface by a triple (Σ_0, h, j) . Two hyperbolic metrics on Σ_0 are conformal equivalent if and only if they are isomorphic to each other. So we can view \mathcal{M}_g as the set of equivalent classes of (Σ_0, h, j) up to isomorphisms, and \mathcal{T}_g as the set of equivalent classes of (Σ_0, h, j) up to isotopic isomorphisms.

Now we will introduce the concept of **Riemann surfaces with nodes**. The precise definition is given in Appendix B.2 of [9]. A compact connected Hausdorff space Σ^* is called a *closed Riemann surface of genus g with nodes* if the following conditions hold. **1°** Every point $p \in \Sigma^*$ either has a neighborhood homeomorphic to $\{z \in \mathbb{C} : |z| < 1\}$ or to $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| < 1, |z_2| < 1\}$, and in the second case we call p a *node*. These complex coordinates give a complex structure j on Σ^* minus nodes. Since Σ^* is compact, there are only finitely many nodes. **2°** Let Σ be Σ^* minus nodes, and $\bar{\Sigma}$ the one point compactification of Σ ¹³. We call Σ the *body* of Σ^* . Every connected component Σ_i of Σ , which we call it a *part* of Σ^* , is of type (g_i, k_i) , which means that Σ_i is gotten by removing k_i distinct points from a Riemann surface of genus g_i , and $2g_i - 2 + k_i > 0$. The second condition makes sure that Σ_i is not homotopic to complex plane and cylinder, which means that Σ_i has the universal cover \mathbb{H} . We call such a part Σ_i having *signature* (g_i, k_i) . **3°** If m and k denote the numbers of nodes and parts of Σ^* , then the genus g is given by $g = \sum_{i=1}^k g_i + m + 1 - k$. The last condition tells us that we can get a Riemann surface Σ_0 of genus g from Σ^* by opening each node.

Two Riemann surfaces with nodes Σ_1^* and Σ_2^* of genus g are said to be *biholomorphically equivalent* if there exists a homeomorphism $f : \Sigma_1^* \rightarrow \Sigma_2^*$ preserving nodes, such that f is biholomorphic between parts $(\Sigma_1)_i$ and $(\Sigma_2)_i$ of Σ_1^* and Σ_2^* respectively. If we add the equivalent classes $[\Sigma^*]$ of Riemann surfaces with nodes of genus g to the moduli space \mathcal{M}_g , we get a compactification $\hat{\mathcal{M}}_g$ of \mathcal{M}_g ¹⁴.

In fact, we are interested in the convergence of $[\Sigma_n] \rightarrow [\Sigma_\infty^*]$ of a sequence of elements in \mathcal{M}_g to the boundary of $\hat{\mathcal{M}}_g$. We will describe the convergence by representing all the equivalent classes by hyperbolic structures. Now let us firstly talk about the hyperbolic representation of Riemann surfaces with nodes. Given a Riemann surface with nodes Σ^* , let j be the complex structure on the body Σ of Σ^* . On each part Σ_i , there

¹³Later on, we will always use Σ^* to denote surface with nodes, Σ surface minus nodes, and $\bar{\Sigma}$ one points compactification of Σ .

¹⁴We refer to Appendix B.2 and B.3 of [9] for topology on $\hat{\mathcal{M}}_g$ and Theorem B.1 of [9] for compactness.

exists a complete hyperbolic metric h compatible with j , with the nodes becoming cusps. So we use (Σ^*, h, j) to denote a *hyperbolic Riemann surface with nodes*. A triple-connected Riemann surfaces with possibly degenerated boundaries is call a *pair of pants*. Fix a hyperbolic Riemman surface with nodes (Σ^*, h, j) , there exists the pair of pants decomposition¹⁵. It means that we can find a largest possible collection of pairwise disjoint, simply closed geodesics $\mathcal{L} = \{\gamma^i : i = 1 \cdots 3g - 3\}$ under the hyperbolic metric h , with γ^i possibly degenerating to nodes, such that each connected component of $\Sigma^* \setminus \mathcal{L}$ is a pair of pants. Now we give a concept for convergence of a sequence of closed hyperbolic Riemann surfaces of genus g to a hyperbolic Riemann surface with nodes¹⁶.

Definition 5.1 *A sequence $\{(\Sigma_n, h_n, j_n)\}$ of closed hyperbolic Riemann surfaces of genus g is said to converge to a hyperbolic Riemann surface with nodes $(\Sigma_\infty^*, h_\infty, j_\infty)$, if there exists a sequence of finite sets $\mathcal{L}_n = \{\gamma_n^i\}_{i=1}^{k_n} \subset \Sigma_n$ constituted by pairwise disjoint simply closed geodesics on (Σ_n, h_n) , with the number of elements k_n bounded by $0 \leq k \leq 3g - 3$, and a sequence of continuous mappings $\phi_n : \Sigma_n \rightarrow \Sigma_\infty^*$, satisfying the following conditions as $n \rightarrow \infty$:*

- 1° : $\phi_n(\gamma_n^i) = p_i$, where p_i is a node on Σ_∞^* , and the length $l(\gamma_n^i) \rightarrow 0$.
- 2° : $\phi_n : \Sigma_n \setminus \mathcal{L}_n \rightarrow \Sigma_\infty$ is a diffeomorphism, where Σ_∞ is the body of Σ_∞^* .
- 3° : $(\phi_n)_* h_n \rightarrow h_\infty$ in $C_{loc}^\infty(\Sigma_\infty)$.
- 4° : $(\phi_n)_* j_n \rightarrow j_\infty$ in $C_{loc}^\infty(\Sigma_\infty)$.

Now using the hyperbolic description of convergence, we can summarize a version of the compactification $\hat{\mathcal{M}}_g$ of \mathcal{M}_g . We refer to Proposition 5.1 of Chap 4 in [8] for a proof.

Proposition 5.1 *For any sequence $\{(\Sigma_n, h_n, j_n)\}_{n=1}^\infty$, where each element (Σ_n, h_n, j_n) represents an equivalent class in \mathcal{M}_g , there exists a subsequence $\{(\Sigma_{n'}, h_{n'}, j_{n'})\}$ converging to a hyperbolic Riemann surface with nodes $(\Sigma_\infty^*, h_\infty, j_\infty)$, which represents an equivalent class in $\hat{\mathcal{M}}_g$.*

Besides the convergence results, we also have a detailed description of the geometry near the degenerating geodesics. We refer to Proposition 4.2 of Chap 4 in [8] and Lemma 4.2 of [16] for the following collar lemma.

¹⁵See Section 3 of [9] and Chap IV. of [8] for detailed discussion of definitions and properties.

¹⁶For general convergence of a sequence of Riemann surfaces with nodes to a fixed Riemann surface with nodes, see Page 71 of [8].

Lemma 5.2 *For any simply closed geodesic γ with length $l(\gamma) = l$ in a hyperbolic surface (Σ, h) , there exists a collar neighborhood of γ , which is isomorphic to the following collar region in hyperbolic plane \mathbb{H} :*

$$\mathcal{C}(\gamma) = \{z = re^{i\theta} \in \mathbb{H} : 1 \leq r \leq e^l, \theta_0(l) \leq \theta \leq \pi - \theta_0(l)\}, \quad (48)$$

with the circles $\{r = 1\}$ and $\{r = e^l\}$ identified by the isometry $\Gamma_l : z \rightarrow e^l z$. Here $\theta_0(l) = \tan^{-1}(\sinh(\frac{l}{2}))$, and γ is isometric to $\{z = re^{\frac{\pi}{2}i} \in i\mathbb{R} : 1 \leq r \leq e^l\}$.

Remark 5.1 *In fact, this result follows from the proof of Lemma 1.6 of Chap 4 in [8]. They consider half of the collar, and they show that the collar region should be part of annuli $\{re^{i\theta} : \theta_0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq y\}$. Instead of using polar coordinates $\{r, \theta\}$, they use the length of boundary of the region $\{re^{i\theta} : \theta_0 \leq \theta \leq \frac{\pi}{2}, r = 1\}$ as parameter. It is easy to change back to polar coordinates and get our formulation above.*

As stated in [15], we can give a explicit metric on the collar region by a conformal change of coordinates. Now, we can view the parameters r and θ in 48 as azimuthal and vertical coordinates for a cylinder respectively. Under the following transformation:

$$r^{i\theta} \rightarrow (t, \phi) = \left(\frac{2\pi}{l}\theta, \frac{2\pi}{l}\log(r)\right),$$

where l is the length of the center geodesic, the collar region $\mathcal{C}(\gamma)$ is changed to a cylinder

$$C = \{(t, \phi) : \frac{2\pi}{l}\theta_0 \leq t \leq \frac{2\pi}{l}(\pi - \theta_0), 0 \leq \phi \leq 2\pi\}, \quad (49)$$

and the hyperbolic metric $ds_{-1}^2 = \frac{|dz|^2}{(Imz)^2}$ is expressed as $ds_{-1}^2 = (\frac{l}{2\pi \sin(\frac{l}{2\pi}t)})^2(dt^2 + d\phi^2)$, which is conformal to the standard cylindrical metric $ds^2 = dt^2 + d\phi^2$. We can see that if the geodesic γ shrink to a point, a conformally infinitely long cylinder will appear.

5.2 Convergence

Before talking about bubbling convergence of the sequence of paths $\{\rho_n(t), \tau_n(t)\}_{n=1}^\infty$ gotten by the previous section, let us firstly clarify the concepts of convergence for a sequence $\{\tau_n\}_{n=1}^\infty \subset \mathcal{T}_g$. Since the area and energy functionals are both conformally invariant, we can choose good representatives in the conformal classes of $\{\tau_n\}_{n=1}^\infty$, or in another word, we would like to project \mathcal{T}_g to \mathcal{M}_g , and use the compactification $\hat{\mathcal{M}}_g$ of \mathcal{M}_g to discuss the convergence of $\{\tau_n\}_{n=1}^\infty$. Here we use hyperbolic representatives as

talked in the above. We say $\{\tau_n\}_{n=1}^\infty$ converge to τ_∞ in $\hat{\mathcal{M}}_g$, if we can find hyperbolic representatives $(\Sigma_n, h_n, j_n) \in \tau_n$ and $(\Sigma_\infty^*, h_\infty, j_\infty) \in \tau_\infty$, such that (Σ_n, h_n, j_n) converge to $(\Sigma_\infty^*, h_\infty, j_\infty)$ in the sense of Definition 5.1. In another word, if we denote $[\tau]$ to be the projection of τ to \mathcal{M}_g , the convergence of $\{\tau_n\}$ to τ_∞ means that $[\tau_n]$ converge to $[\tau_\infty]$ in $\hat{\mathcal{M}}_g$. Now we can state the following theorem.

Theorem 5.1 *Let $\{(\rho_n(t), \tau_n(t))\}_{n=1}^\infty$ be the sequence gotten by the perturbation from $\{(\gamma_n(t), \tau_n(t))\}_{n=1}^\infty$ by Lemma 4.1¹⁷, then all min-max sequences $\{(\rho_n(t_n), \tau_n(t_n))\}_{n=1}^\infty$ with $E(\rho_n(t_n), \tau_n(t_n)) \rightarrow \mathcal{W}_E$, satisfy:*

() For any finite collection of disjoint geodesic balls $\bigcup_i B_i$ on $\Sigma_{\tau_n(t_n)}$ with radii bounded as in Lemma 4.1, such that $E(\rho_n(t_n), \bigcup_i B_i) \leq \epsilon_0$, let v be the harmonic replacement of $\rho_n(t_n)$ on $\frac{1}{64} \bigcup_i B_i$. We have:*

$$\int_{\frac{1}{64} \bigcup_i B_i} |\nabla \rho_n(t_n) - \nabla v|^2 \rightarrow 0 \quad (50)$$

By compactness Proposition 5.1, a subsequence of $\{\tau_n(t_n)\}_{n=1}^\infty$ converge to some τ_∞ in $\hat{\mathcal{M}}_g$ up to a subsequence, which is achieved by the convergence of a sequence hyperbolic Riemann surfaces $(\Sigma_n, h_n, j_n) \in \tau_n(t_n)$ to $(\Sigma_\infty^*, h_\infty, j_\infty) \in \tau_\infty$ as in Definition 5.1. If we denote the one point compactification of Σ_∞ by $\bar{\Sigma}_\infty$, and j_∞ the extended complex structure, then there exist a conformal harmonic map $u_0 : (\bar{\Sigma}_\infty, j_\infty) \rightarrow N$ and possibly some harmonic spheres $\{u_i : S^2 \rightarrow N \mid i = 1, \dots, l\}$, such that $(\rho_n(t_n), (\Sigma_n, h_n, j_n))$ bubbling converge¹⁸ to (u_0, u_1, \dots, u_l) , with:

$$\lim_{n \rightarrow \infty} E(\rho_n(t_n), j_n) = E(u_0, j_\infty) + \sum_i E(u_i) \quad (51)$$

Remark 5.2 *In fact, property (*) in the above theorem is scaling invariant, so we can apply the Sacks-Uhlenbeck bubbling convergence theory to $\{\rho_n(t_n)\}$. In fact, the left hand side of 51 is the min-max critical value \mathcal{W} , and the right side is the sum of areas since (u_0, u_1, \dots, u_l) are all conformal, so we get the conclusion that the min-max critical value is achieved by the area of a set of minimal surfaces.*

The proof is divided into several steps in the following sections.

¹⁷See the discussion in the beginning of this section to see how to achieve the no non-constant harmonic slice condition.

¹⁸See [12][13] and Appendix B.6 in [5] for more details about bubbling convergence.

5.2.1 Convergence on domains

Firstly we summarize the known facts of convergence of almost harmonic maps defined on a sequence of converging domains. Suppose $\{(\Omega_n, h_n, j_n)\}_{n=1}^\infty$ is a sequence of two dimensional domains with metric h_n and compatible complex structure j_n . We assume that $(\Omega_n, h_n, j_n) \rightarrow (\Omega_\infty, h_\infty, j_\infty)$ in the following sense. For n large enough, there exist a sequence of diffeomorphisms $\phi_n : \Omega_\infty \rightarrow \Omega_n$, such that the pull-back metrics and complex structures converge, i.e. $(\phi_n)^* h_n \rightarrow h_\infty$ and $(\phi_n)^* j_n \rightarrow j_\infty$ in C^3 on any compact subsets of Ω_∞ . Let $\{u_n : (\Omega_n, h_n, j_n) \rightarrow N\}_{n=1}^\infty$ be a sequence of $W^{1,2}$ almost harmonic maps satisfying the following condition:

(*1) : For any geodesic small $B \in \Omega_n$ with radius smaller than the infimum¹⁹ of injective radii of all (Ω_n, h_n) and r_0 as in (22), if $E(u_n, B) < \epsilon_0$ with ϵ_0 ²⁰ given by Lemma 4.1, denote v to be the harmonic replacement of u_n on $\frac{1}{64}B$, then:

$$\int_{\frac{1}{64}B} |\nabla u_n - \nabla v|^2 \leq \delta(n) \rightarrow 0.$$

Lemma 5.3 *For a sequence $\{u_n : (\Omega_n, h_n, j_n) \rightarrow N\}_{n=1}^\infty$ as above with $E(u_n, j_n) \leq E_0 < \infty$, there exist finitely many points $\{x_1, \dots, x_k\} \subset \Omega_\infty$, a subsequence $\{n'\}$ and a harmonic mapping $u_\infty \in W^{1,2}(\Omega \setminus \{x_1, \dots, x_k\}, N)$, such that for any compact subset $K \subset \Omega_\infty \setminus \{x_1, \dots, x_k\}$, the subsequence $u_{n'} : (\phi_{n'}(K) \subset \Omega_{n'}, h_{n'}, j_{n'}) \rightarrow N$ converge in $W^{1,2}$ to u_∞ .*

Remark 5.3 *The convergence of $u_{n'}$ to u_∞ can be understood as the convergence after pulling $u_{n'}$ back to Ω_∞ by $\phi_{n'}$. We call points $\{x_1, \dots, x_k\}$ energy concentration points. The proof of results similar to the above lemma is given in [12][13], Appendix B.2 of [5] and the proof of Theorem 5.1 of [15]. In fact, step 1 of the proof of Theorem 5.1 in [15] almost directly gives the proof of the above lemma, so we omit it. By the Removable Singularity Theorem 3.6 of [12], we can extend u_∞ to a harmonic map on Ω_∞ .*

5.2.2 Convergence on cylinders

Now based on the above lemma, the next step to study the convergence of $\{(\rho_n, \tau_n)\}_{n=1}^\infty$ is to do rescaling near energy concentration points, and to consider regions near degenerating geodesics. In both of the cases which we will discuss in detail later, we need to

¹⁹The infimum exists and is positive because of the convergence.

²⁰In order to apply Sacks-Uhlenbeck bubbling convergence, we can pick $\epsilon_0 < \epsilon_{SU}$, where ϵ_{SU} is a small constant depending only on the ambient manifold N given in [12].

consider almost harmonic maps on long cylinders. We use $\mathcal{C}_{t^1, t^2} = \{(t, \theta) \in \mathbb{R} \times S^1 : t^1 \leq t \leq t^2, \theta \in [0, 2\pi)\}$ to denote a cylinder with length parameter between t^1 and t^2 , and h a metric on \mathcal{C}_{t^1, t^2} conformal to the standard metric $ds^2 = dt^2 + d\theta^2$. We denote $S_{t^0} = \{(t, \theta) : t = t^0, \theta \in [0, 2\pi)\}$ to be a slice of \mathcal{C}_{t^1, t^2} . We say a sequence of cylinders $\{(\mathcal{C}_{t_n^1, t_n^2}, h_n) : 1 \leq n < \infty\}$ converge to $(\mathcal{C}_\infty = \mathbb{R} \times S^1, ds^2 = dt^2 + d\theta^2)$, if when we identify all the cylinders by the center slices $S_{t_n^0}$ with $t_n^0 = \frac{1}{2}(t_n^1 + t_n^2)$, the metrics h_n converges in C^3 to ds^2 on any compact subsets of \mathcal{C}_∞ , i.e. when we choose $\phi_n : \mathcal{C}_{t_n^1, t_n^2} \rightarrow \mathcal{C}_\infty$, such that $\phi_n(t, \theta) = (t - t_n^0, \theta)$, then $(\phi_n)_* h_n \rightarrow ds^2$ in $C^3(K)$ for any compact subset $K \subset \mathcal{C}_\infty$. Consider a sequence of almost harmonic maps defined on a sequence of converging cylinders $\{u_n : (\mathcal{C}_{t_n^1, t_n^2}, h_n) \rightarrow N \mid n = 1, \dots, \infty\}$ satisfying property (*1) in the above section. By Lemma 5.3, they sub-converge to a harmonic map on \mathcal{C}_∞ . Before talking about further results, we need to introduce another type of almost harmonic maps and a corresponding energy estimate.

Definition 5.2 For $\nu > 0$, we call $u \in W^{1,2}((\mathcal{C}_{r_1, r_2}, h), N)$ a ν -**almost harmonic map** (Definition B.27 in [5]) if for any finite collection of disjoint geodesic balls \mathcal{B} in $(\mathcal{C}_{r_1, r_2}, h)$ with radii bounded from above by the injective radius of $(\mathcal{C}_{r_1, r_2}, h)$ and r_0 as in (22), there is an energy minimizing map $v : \frac{1}{64}\mathcal{B} \rightarrow N$ with the same boundary value as u such that:

$$\int_{\frac{1}{64}\mathcal{B}} |\nabla u - \nabla v|^2 \leq \nu \int_{\mathcal{C}_{r_1, r_2}} |\nabla u|^2. \quad (52)$$

This definition traces back to Definition B.27 in [5], but we modify it here to be adapted to our setting. Now a proof similar to that of Proposition B.29 of [5] gives a similar estimate as follows.

Proposition 5.2 $\forall \delta > 0$, there exist small constants $\nu > 0$ (depending on h , δ and N), $\epsilon_2 > 0$ and large constant $l \geq 1$ (depending on δ and N), such that for any integer m , if u is a ν -almost harmonic map defined above on $(\mathcal{C}_{-(m+3)l, 3l}, h)$ with $E(u) \leq \epsilon_2^{21}$, then:

$$\int_{\mathcal{C}_{-ml, 0}} |u_\theta|^2 \leq 7\delta \int_{\mathcal{C}_{-(m+3)l, 3l}} |\nabla u|^2. \quad (53)$$

Here u_θ means the differentiation w.r.t θ .

Now we would like to give a more precise description of the convergence on cylinders.

²¹We can let $\epsilon_2 < \epsilon_{SU}$ as above again.

Lemma 5.4 *In the convergence of $u_n : (\mathcal{C}_{t_n^1, t_n^2}, h_n) \rightarrow N$ as discussed above, if $E(u_n) \leq \epsilon_2$ with ϵ_2 given in the above proposition, then either $\liminf_{n \rightarrow \infty} E(u_n) = 0$, or u_n must be uniformly un-conformal for n large enough in the following sense, i.e. there exists a small number δ_0 , such that:*

$$E(u_n) - \text{Area}(u_n) \geq \delta_0. \quad (54)$$

Furthermore, if $\{u_n\}$ are almost conformal and $\liminf_{n \rightarrow \infty} E(u_n) \geq \epsilon_2$, then there exists a large fixed number $L > 0$, such that $E(\rho_n, \mathcal{C}_{r_n^0 - L, r_n^0 + L}) \geq \epsilon_2$, i.e. the energy must concentrate on some finite part of the cylinders.

Remark 5.4 *This is a summarization of the results proved in step 5 of the proof of Theorem 5.1 in [15]. In fact, if $E(u_n) \leq \epsilon_2$ and $\liminf_{n \rightarrow \infty} E(u_n) > 0$, it is easy to show that u_n is μ -almost harmonic as in Definition 5.2 for μ small enough when n is large enough. If we apply the estimate in Proposition 5.2, we get an upper bound for $\int_{\mathcal{C}_{-m, 0}} |(u_n)_\theta|^2$. Then by computing the difference between the energy and area of u_n as in equation 55 of [15], we will get the lower bound for $E(u_n) - \text{Area}(u_n)$. In the second case, we use contradiction argument. We will go back to the first case to get a sequence of almost harmonic mappings on long cylinders with energy bounded from above by ϵ_2 and away from 0, which will lead to a contradiction to almost conformal property. We omit the detailed proof here and refer that to [15].*

5.2.3 Proof of Theorem 5.1

Now we use the results summarized above to show the bubble convergence and energy identity 51 of Theorem 5.1. Let us denote $\rho_n = \rho_n(t_n)$, and $\tau_n = \tau_n(t_n)$ in the following.

Step 1: bubble convergence on domain surfaces. In the convergence of $(\Sigma_n, h_n, j_n) \in \tau_n$ to $(\Sigma_\infty^*, h_\infty, j_\infty) \in \tau_\infty$, let us denote \mathcal{L}_n to be the sets of geodesics and $\phi_n : \Sigma_n \rightarrow \Sigma_\infty^*$ the continuous mappings as in Definition 5.1. Now let us consider the sequence of almost harmonic maps $\{\rho_n : (\Sigma_n \setminus \mathcal{L}_n, h_n, j_n) \rightarrow N\}_{n=1}^\infty$ satisfying property (*) in Theorem 5.1. By Lemma 5.3, there exists a finite set of energy concentration points $\{x_1, \dots, x_l\}$ on the body Σ_∞ of Σ_∞^* , and a subsequence which we still denote by ρ_n , that converge to a harmonic map $u_0 : \Sigma_\infty \rightarrow N$ in $W^{1,2}$ on any compact subsets of $\Sigma_n \setminus (\mathcal{L}_n \cup \phi_n^{-1}\{x_1, \dots, x_l\})$. Denote $x_{n,i} = \phi_n^{-1}(x_i)$. Near each energy concentration point $x_{n,i}$, let $r_{n,i}$ be the smallest radii such that $E(\rho_n, B_{x_{n,i}, r_{n,i}}) = \epsilon_0$ with ϵ_0 as in condition (*1), where $B_{x_{n,i}, r}$ denotes the hyperbolic geodesic balls centered at $x_{n,i}$ with radii

r on Σ_n . View $B_{x_{n,i}, r_{n,i}}$ as a ball on the Poincaré disk (D, ds_{-1}^2) centered at the origin 0, and use the coordinates there. Now rescale $B_{x_{n,i}, r_{n,i}}$ to $B_{0,1} \subset \mathbb{C}$ by $x \rightarrow x/r_{n,i}^0$, where $r_{n,i}^0$ is the Euclidean radius of $B_{x_{n,i}, r_{n,i}}$ measured w.r.t. the Euclidean metric on (D, ds_0^2) . In fact, $r_{n,i}$ and $r_{n,i}^0$ are almost the same when $r_{n,i} \rightarrow 0$ as $n \rightarrow \infty$. Then rescale the hyperbolic metric ds_{-1}^2 to be $ds_n^2 = \frac{|dz|^2}{1-|r_{n,i}^0 z|^2}$, which converge to the flat metric on any compact subsets of \mathbb{C} . Let $u_{n,i}(x) = \rho_n(r_{n,i}^0 x)$. Since properties $(*)$ and $(*1)$ are scaling invariant, the sequence $\{(u_{n,i}, (B_{r/r_{n,i}^0}, ds_n^2))\}_{n=1}^\infty$ satisfy the requirement of Lemma 5.3 again for some fixed small radius r . So a subsequence of $\{u_{n,i}\}_{n=1}^\infty$ converge in $W^{1,2}$ to a harmonic map $u_{\infty,i}$ defined on \mathbb{C} in the sense of Lemma 5.3 again. We can repeat such processes near energy concentration points step by step. An important observation is that $u_{\infty,i} : \mathbb{C} \rightarrow N$ is a nontrivial harmonic map, since the energy of $u_{n,i}$ over $B(0,1)$ is ϵ_1 by the conformal invariance of energy and our choice of the bubbling region $B_{x_{n,i}, r_{n,i}}$, and $u_{\infty,i}$ extends to a harmonic map on the sphere, whose energy is bounded below by ϵ_{SU} (see [12]). We call all such harmonic spheres bubbles. So for each step, the total energy is decreased by some fixed amount, hence it must stop in finitely many steps.

Step 2: bubble convergence on necks and collars. To prove the energy identity 51, we need to study the behavior of the limit process on some small annuli and collar neighborhoods of degenerating geodesics. Near an energy concentration point, if we compare the energy limit $\lim_{n \rightarrow \infty} E(\rho_n, B(x_i, r))$ with the sum of the limit energy $E(u_0, B(x_i, r))$ and bubble energy $\lim_{n \rightarrow \infty} E(u_{n,i}, B_{r/r_{n,i}^0})$, we need to count the neck part, which is $\lim_{r \rightarrow 0, R \rightarrow \infty} \lim_{n \rightarrow \infty} E(\rho_n, B(x_i, r) \setminus B(x_i, r_{n,i}^0 R))$. Here we refer to the step 4 of proof of Theorem 5.1 in [15] for details. Denote the annuli by $A(x_i, r, r_{n,i}^0 R) = B(x_i, r) \setminus B(x_i, r_{n,i}^0 R)$, and we call them necks. Under the change of coordinates $(r, \theta) \rightarrow (t, \theta) = (\log r, \theta)$, the annuli are changed to long cylinders $\mathcal{C}_{r_n^1, r_n^2}$, with $r_n^1 = \ln(r_{n,i} R)$, $r_n^2 = \ln(r)$, and the hyperbolic metrics are $ds_{-1}^2 = \frac{e^{2t}}{1-e^{2t}}(dt^2 + d\theta^2)$. When we rescale the metrics such that the center slice $S_{t_n^0}$ has length 2π , it is easy to see that the metrics converge to the flat metric on any compact subset of the infinite long cylinder $\mathbb{R} \times S^1$. Since property $(*)$ is invariant under scaling, we go back to the setting for the previous section. We will continue studying the convergence in this case after we introducing the behavior near degenerating geodesics.

Now let us see the behavior near degenerating geodesics $\gamma_n^i \in \mathcal{L}_n$. Similar arguments as in the case of necks show that if we want to recover all the energy of ρ_n on Σ_n from the limit u_0 and all the bubbles u_n , we need to consider the amount of energy on the collar neighborhoods $\mathcal{C}(\gamma_n^i)$ given by Lemma 5.2. As in equation 48, we use (r, θ) as parameters

for the cylinder, and denote $\mathcal{C}(\gamma_n^i, \theta_0)$ to be the sub-collar with $\theta_0 \leq \theta \leq \pi - \theta_0$. In fact, as $l_n = l(\gamma_n^i) \rightarrow 0$, we need to take care of the limit $\lim_{\theta_0 \rightarrow \frac{\pi}{2}} \lim_{n \rightarrow \infty} E(\rho_n, \mathcal{C}(\gamma_n^i, \theta_0))$. Using the change of coordinates as given in the remark below Lemma 5.2, those collars can be viewed as a sequence of cylinders $\mathcal{C}_{r_n^1, r_n^2}$ with $r_n^1 = \frac{2\pi}{l_n} \theta_0$, $r_n^2 = \frac{2\pi}{l_n} (\pi - \theta_0)$. If we rescale the hyperbolic metrics $ds_{-1}^2 = (\frac{l_n}{2\pi \sin(\frac{l_n}{2\pi} t)})^2 (dt^2 + d\phi^2)$ on $\mathcal{C}_{r_n^1, r_n^2}$ such that the center slice $S_{(\frac{2\pi}{l_n})\frac{\pi}{2}}$ has length 2π , it is easy to see that those metrics converge to the flat metric on any compact subset of $\mathbb{R} \times S^1$, which goes back to the setting for the previous section again by the conformal invariance of property (*).

Summarizing the above two paragraphs, we need to study the case of a sequence of almost harmonic maps defining on cylinders approximating the infinite long standard cylinder. If $\liminf_{n \rightarrow \infty} E(\rho_n, (\mathcal{C}_{r_n^1, r_n^2}, ds_{-1}^2)) = 0$, then we can discard this part in the energy identity 51, or since our sequence of maps are almost conformal by Lemma 5.1, $\liminf_n E(\rho_n, (\mathcal{C}_{r_n^1, r_n^2}, ds_{-1}^2)) \geq \epsilon_2$ by Lemma 5.4. Then there exists a large fixed number $L > 0$, such that $E(\rho_n, \mathcal{C}_{r_n^0 - L, r_n^0 + L}) \geq \epsilon_2$ by Lemma 5.4 again. Now $(\rho_n, (\mathcal{C}_{r_n^1, r_n^2}, ds_{-1}^2))$ converge in $W^{1,2}$ to a harmonic map $u_\infty : \mathbb{R} \times S^2 \rightarrow N$ on any compact subsets of $\mathbb{R} \times S^2$ minus possibly finite many energy concentration points by Lemma 5.3. We can repeat the above steps near energy concentration points again. Now in order to count all the energy, we need to consider sub-cylinders $\mathcal{C}_{t_n - L_n, t_n + L_n} \subset \mathcal{C}_{r_n^1, r_n^2}$ with $|t_n - t_n^0| \rightarrow \infty$ and $L_n \rightarrow \infty$. We need to show that $\lim_{n \rightarrow \infty} E(\rho_n, \mathcal{C}_{t_n - L_n, t_n + L_n})$ is counted by some bubble maps. In fact, when we rescale the metrics such that the center slice S_{t_n} of $\mathcal{C}_{t_n - L_n, t_n + L_n}$ has length 2π , the sequence of cylinders will converge to $\mathbb{R} \times S^1$ again as in the previous section. So we can repeat the steps again.

We can see that no energy loss will happen since once there are energy concentrated on long cylinders, they must be counted in the next bubbling step. We know that either $u_\infty : \mathbb{R} \times S^1 \rightarrow N$ is nontrivial, which can be extended to a harmonic map on S^2 by removable singularity theorem in [12], since S^2 is conformal to $\mathbb{R} \times S^1$, or some of the bubble maps near energy concentration points are nontrivial since $E(\rho_n, \mathcal{C}_{r_n^0 - L, r_n^0 + L}) \geq \epsilon_2$. So each of such steps also takes away a fixed amount of energy, so we must stop in finite many steps. All such steps form the convergence in Theorem 5.1. Count all the energy of those finite many bubble maps, which are harmonic maps on spheres, we will get the energy identity 51. So we finish the proof.

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